International School of Physics

Statistical tools for time varying quantities

Patrizia Tavella
Istituto Nazionale Ricerca Metrologica
Torino Italy

Metrology: from physics fundamentals to quality of life
July 2016
Repeated measurements reveal always something

- stationarity
- aging
- slow drift
- abrupt change
- failure

Around $t = 500$, the noise increases.
Changes may cause disturbance: the mass of the International Prototype Kilogram

Masses of the entire worldwide ensemble of prototypes have been slowly diverging from each other.
Or may reveal new physics

Or just give the complete picture

LIGO Sees First Gravitational Waves
Statistical and Mathematical Tools

- to estimate the behaviour of a physical quantity and its possible impact on complex systems
- to predict the behaviour
- to control the behaviour

- noise filtering
- long term analysis
- uncertainty propagation
- identifying drift
- data validation
- calibration intervals
- missing data reconstruction

The uncertainty on the single measure can be evaluated according to the GUM: Guide for Uncertainty in Measurement.

instability is more impacting than single measure uncertainty
Instability is more impacting than single measure uncertainty

Instability can be due to the measurement process, to the reference standard, to device under test, or to the measurand

- If we want to produce and sell a reference measurement standard or measuring device
- If we want to use the standard as primary reference for calibrating other devices
- If we want to carry out precise measure to test fundamental physics
- If we use the standard or the measurement device in a complex system, as space, health, environment applications
Time series of measures may contain polynomial component to be treated by least square estimation techniques (batch, recursive...)

\[ y(t) \]

- Linear drift
- Quadratic change
- Initial offset

\[ t \]
At each instant the measure $y(t)$ is a random variable,

but stochastic components are also present and sometime dominant.
Appropriate mathematical tools for describing the stochastic noises:

- Spectral analysis of noise
- Filtered variances such as the Allan variance

\[ f \]

decomposition in elementary frequencies

dispersion of the average frequency values
The variance is the measure of dispersion

\[
\sigma^2(\tau) = \frac{\sum_{i=1}^{N} (\bar{y}_i - \mu)^2}{N}
\]

First case of typical noisy behaviour (short term noise):

If the “noise” is stationary, the estimation of the variance converges as \( N \) grows

For longer observation interval \( \tau \), the variability diminishes

The “noise” is white and

\[
\sigma^2(\tau) \propto \frac{1}{\tau}
\]

The variance depends on the observation interval \( \tau \)

Varenna 2016
Instability - dispersion - variability

The variance is the measure of dispersion:

\[ \sigma^2(\tau) = \frac{\sum_{i=1}^{N} (\bar{y}_i - \mu)^2}{N} \]

Other typical noisy behaviour (long term noise)

If the “noise” is not strictly stationary, e.g. a random walk, the estimation of the variance do not converge and depends on \( N \) the number of samples (for any \( \tau \))
If the “noise” is not strictly stationary, e.g. a random walk, the estimation of the variance do not converge and depends on $N$ the number of samples (for any $\tau$)

IDEA of Allan and Barnes (1966) let’s agree on the number of samples $N=2$

$$\sigma^2(N = 2, \tau) = \frac{\sum_{i=1}^{2} (\bar{y}_i - \mu)^2}{2} = \frac{1}{2} (\bar{y}_1 - \bar{y}_2)^2$$

and let’s average many 2-sample variances

$$\sigma_y^2(\tau) = \frac{1}{2} \left\langle (\bar{y}_{t+\tau} - \bar{y}_t)^2 \right\rangle$$

Allan variance

stability of the mean values (on $\tau$ intervals), actually variance of increments


Varenna 2016
With the Allan deviation we understand which type of noise is (mostly) affecting the measures, usually applied to the frequency of the atomic clocks.

\[ \tau^{-1/2} = \text{white freq noise} \]

\[ \tau^{+1/2} = \text{random walk freq noise} \]
Allan variance ↔ spectral density

\[ S_y(f) = h_\alpha(f)^\alpha \]

\[ \log S_y(f) \]

\[ \log \sigma^2(\tau) \propto (\tau)^\mu \]

\[ <\sigma^2(\tau)> \propto (\tau)^\mu \]

\[ \log <\sigma^2(\tau)> \]

\[ \log \sigma^2(\tau) \]

\[ \text{random walk frequency} \]

\[ \text{flicker frequency} \]

\[ \text{white frequency} \]

\[ \alpha = -2 \]

\[ \alpha = -1 \]

\[ \alpha = 0 \]

\[ \alpha = 1 \]

\[ \alpha = 2 \]

\[ \text{white noise} \]

\[ \text{flicker phase} \]

\[ \text{flicker frequency} \]

\[ \text{white frequency} \]

\[ \mu = -2 \]

\[ \mu = -1 \]

\[ \mu = 0 \]

\[ \mu = 1 \]

\[ \text{random walk frequency} \]

\[ \text{observation interval } \tau \]
1. In some cases, the frequency instability is more impacting than single measure uncertainty.

2. The instability variance depends on the observation interval $\tau$.

3. In some cases, the noise is non strictly stationary and the classical variance is not an appropriate tool. The Allan variance was proposed and important properties were found.
Most common noise are white, random walk, and integrated random walk.

Time offset = Integral of Frequency offset

Allan Deviation

\[ \tau^{-1/2} = \text{white noise} \]

\[ \tau^{+1/2} = \text{random walk noise} \]
The mathematical model for deterministic and stochastic behaviour
A mathematical model for determinist and stochastic behaviour

White noise plus constant offset and its integrated effect

Velocity

\[ y(t) = y_0 + \xi(t) \]

Position

\[ x(t) = x_0 + y_0(t - t_0) + \int_{t_0}^{t} \xi(s) ds \]

**Velocity**

**Position**

\[ \xi(t) \sim N(0, \sigma^2) \]

White Frequency Noise + constant initial offset

Straight Line + Random Walk on Time Offset

Varenna 2016
Mathematical context: notations

Random Walk

Wiener Process $W(t)$ (Brownian Motion)

White Noise $\xi(t)$

"derivative" of a Wiener process

$\xi(t)dt = dW(t)$
\[ \xi(t)dt = dW(t) \]

**Stochastic calculus**

\[ dx(t) = y_0 dt + \sigma dW(t) \]

\[ x(t) = x_0 + y_0(t - t_0) + \int_{t_0}^{t} \xi(s)ds \]

\[ y(t) = \frac{dx(t)}{dt} = y_0 + \xi(t) \]

It depends on the choice of \( t_i \)

\[ \approx \lim_{\delta \to 0} \sum_{i=1}^{n} (W(t_i) - W(t_{i-1})) \]

\( \delta = t_i - t_{i-1} \)

**Itô stochastic integration**

*White is the derivative of Wiener or Random Walk*
Integrated (White noise plus Random Walk noise)

\[
dx(t) = y_0 dt + \sigma \ dW(t)
\]

\[
\begin{aligned}
x_1(t) &= x_2(t) dt + \sigma_1 \ dW_1(t) \\
x_2(t) &= a \ dt + \sigma_2 \ dW_2(t)
\end{aligned}
\]

with initial conditions
\[
\begin{aligned}
x_1(0) &= x_0 \\
x_2(0) &= y_0
\end{aligned}
\]

\(x_1(t) = \text{time offset} = \text{integrated effect as position}\)

\(x_2(t) = \text{a component of frequency offset or of velocity}\)
In case of integrated white and random walk the exact solution exists:

\[
\begin{align*}
    x_1(t) &= x_0 + y_0 t + a \frac{t^2}{2} + \sigma_1 W_1(t) + \sigma_2 \int_0^t W_2(s) ds \\
    x_2(t) &= y_0 + a t + \sigma_2 W_2(t)
\end{align*}
\]

Iterative solution useful for simulations, filter, ...

\[
\begin{align*}
    X_1(t_{k+1}) &= X_1(t_k) + X_2(t_k) \tau + a \frac{\tau^2}{2} + \sigma_1 W_{1,k}(\tau) + \sigma_2 \int_{t_k}^{t_{k+1}} W_2(s) ds \\
    X_2(t_{k+1}) &= X_2(t_k) + a \tau + \sigma_2 W_{2,k}(\tau)
\end{align*}
\]

with initial conditions

\[
\begin{align*}
    X_1(0) &= x_0 \\
    X_2(0) &= y_0
\end{align*}
\]
Random Walk prediction

\[ x_0 = y_0 = a = 0 \]

The prediction error at epoch \( t \) can be evaluated

\[ \sigma = \sqrt{\sigma^2 t} \]

Diffusion coefficient linked to Allan Deviation

Example of a Cesium clock, 10 days after synchronisation


Other stochastic process may be useful as the **Ornstein–Uhlenbeck** process

The O-U process is the solution of the following sde

\[
dU_t = -\frac{1}{\tau} U_t \, dt + \sigma \, dW_t.
\]

the parameter \( \sigma \) measures how much *noisy* is the process. For \( \tau \):

**Exact iterative formula (\( U \) at discrete times \( t_{n+1} = t_n + h \))**

\[
U_n = U_{n-1} \cdot e^{-\frac{h}{\tau}} + \sqrt{\frac{\sigma^2 \tau}{2} \left(1 - e^{-\frac{2h}{\tau}}\right)} \xi_k
\]

where \( \{\xi_k\} \) are i.i.d. standard normal r.v.
Other stochastic process may be useful as the **Ornstein–Uhlenbeck** process.

Limit behaviors:

- **O-U** w.r.t. Brownian Motion
  \[ (\tau \to \infty) \]
  e.g. phase data of a white of frequency

- **O-U** w.r.t. White Noise
  \[ (\tau \to 0) \]

---

1. Deterministic and stochastic behaviour can be modeled by stochastic differential equations (SDE).

2. The exact (or approximat) solution of the SDE allows the dynamic behaviour estimation, simulation, and prediction.

3. The noises imbedded in the model has usually zero mean value, they do not impact on the prediction but on the uncertainty of the predicted values (therefore allowing to evaluate confidence intervals).
Will the well understood and modeled behaviour last forever?
Example of GPS space clocks

Possible causes of nonstationarities on space clocks are manoeuvres and tests on board, environmental variations, eclipses, etc. In other cases, the nonstationarities may be due to the clock itself.

Frequency jumps are observed
WE NEED A FREQUENCY JUMP DETECTOR


From the past we best predict the future and then we compare prediction with measures (or mean of measures)

Nw = full data set
Nw₁ = estimation of deterministic behaviour
Nw₂ = check on jump
Nw₃ = moving average to smooth noise

\( \hat{\mu} \) = estimated value of deterministic trend
\( \hat{\mu}_t \) = extrapolated deterministic trend
yₛ = frequency values smoothed by a moving average on Nw₃ samples (sliding in window Nw₂)

\[ \Delta y = | y_s - \hat{\mu}_t | \]

If \( \Delta y > \) threshold

Alarm

\( y \)

\( \hat{\mu} \)

\( \hat{\mu}_e \)
QUICKEST DETECTION METHOD (optimal stopping) FOR A WIENER PROCESS

Assume that the quantity evolution can be modeled by a Wiener process $X$ and observe a trajectory with a drift changing from 0 to $\mu \neq 0$ at some random time $\theta$.

Task: find a stopping time $\tau$ of $X$ that is as close as possible to the unknown time $\theta$.

QUICKEST DETECTION METHOD (optimal stopping)

FOR A WIENER PROCESS

We write a risk function

$$V(\pi) = \inf_{\tau} \left( P(\tau < \theta) + cE[\tau - \theta]^+ \right)$$

- **Probability of false alarm**: $P(\tau < \theta)$
- **Average delay in detecting the anomaly**: $E[\tau - \theta]^+$
- **A suitable constant**: $c$

We want $P(\tau < \theta)$ and also $E[\tau - \theta]^+$ to be small

- To avoid false alarms
- To minimize the detection delay
The minimization problem

\[ V(\pi) = \inf_{\tau} \left( P(\tau < \theta) + c E[\tau - \theta]^+ \right) \]

can be written as an optimal stopping problem

\[ V(\pi) = \inf_{\tau} E \left[ 1 - \pi_{\tau} + c \int_0^\tau \pi_t \, dt \right] \]

where \( \pi_t = P_\pi(\theta \leq t | \mathcal{F}_t^X) \) with \( P_\pi(\pi_0 = \pi) = 1 \)

is the a posteriori probability process, probability that by epoch \( t \) the process \( X \) has changed drift.

The original process \( X \) has changed drift when the process \( \pi \) is crossing the boundary \( A \).
The optimization problem has been transformed into a first passage time, becoming an analytical problem.
A theorem demonstrates that the exact solution exists and studying the additional process $\pi$ we can optimally estimate the epoch of the insurgence of the new drift $\mu$.

$$\star = \theta = 25$$

$$\diamond = \tau = 26.63$$

$\mu = 3, \sigma^2 = 1$. 
We apply the optimal stopping with different values of $E(\theta) = 1/\lambda$.

- $E(\theta) = 3 \times 10^7$ (one event a year)
  - $\tau = 5087.5$

- $E(\theta) = 10^6$ (one event every 28 days)
  - $\tau = 5087.5$

- $E(\theta) = 1500$ (one event every 25 min)
  - $\tau = 5086.5$
The NASA New Horizon spacecraft was launched on Jan 19th, 2006 to meet Pluto on July 14, 2015. The measurement system required an ultra-stable oscillator (Quartz Crystal) with very high frequency stability.

Example of optimal stopping

The gaps in the data are those periods when the spacecraft was in hibernation and no tracking was performed.

During 2015, the frequency of the Quartz was measured almost continuously in preparation for the Pluto-Charon encounter.

A reversal in the frequency rate asks for recovery actions.
Will the well understood and modeled behaviour last forever?
We need a **dynamical characterization of the noise**

*Time and Frequency spectral analysis* for example

Not only estimating *which* frequencies existed

But also estimating *when* they existed
Time-frequency analysis

It describes how the frequencies of a signal change with time

A Dynamic Allan variance

sliding the Allan variance estimator on the data

\[ \sigma_y(t_1, \tau) \quad \sigma_y(t_2, \tau) \quad \ldots \quad \sigma_y(t, \tau) \]
stability may vary with time

stable sliding estimator

Simulation results: Bump

\[ \sigma_y(t, \tau) \]

\[ \hat{\sigma}_y(t, \tau) \]
The Dynamic Allan variance

Discrete time formulation from the phase samples $x[n]$

$$
\sigma^2_y[n, k] = \frac{1}{2k^2 \tau_0^2} \frac{1}{N_w - 2k} \sum_{m=n-N_w/2+k}^{n+N_w-2-k} E\left[(x_N[m+k] - 2x_N[m] + x_N[m-k])^2\right]
$$

where:

- $N_w$ is the window length
- $x_N$ is the phase signal in the window $N_w$
- $\tau_0$ is the sampling time

the DAVAR estimator

has no expectation value $E$ because we have one realization only


Varennna 2016
The change in instability is easily detected

CANVAS by NRL @ https://goby.nrl.navy.mil/canvas/download/
STABLE 32 Users: Upgrade to version 1.5
We insert a “threshold” surface to detect increase of instability

Reference surface = nominal clock

\[ \sigma_H(t, \tau) = \gamma \tau^{1/2} \]
The threshold surface may reveal a noise increasing in time
Demonstrating stationarity

DADEV of a new space Rubidium atomic frequency standard, the Robust-RAFS, developed by Orolia Switzerland SA (SpectraTime). The clock manufacturer demonstrates to the customer that the Robust-RAFS follows the specifications throughout the entire performance test.

(courtesy of Fabien Droz)
Application to cardiology

DADEV of the heart interbeat rate for a normal patient

courtesy of Ricardo Hernández-Pérez

DADEV of the heart interbeat rate for a patient suffering from CHF (congestive heart failure)


Varenna 2016
1. The instability of time varying quantities may be highly impacting

2. Instability may be estimated by appropriate tools mathematical models including noises can be written

3. Models allow estimation, prediction, simulation, control

4. The behaviour may change due to ageing, failures, wearing... These changes are to be detected (rapidly) and the model is be dynamically updated

5. How many other statistical tools are useful in Metrology?

Thank you for your attention