Electrical Standards based on quantum effects: Part II
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Part II: The Quantum Hall Effect
Overview

- Classical Hall effect
- Two-dimensional electron gas
- Landau levels
- Measurement technique
- Accuracy of the quantized Hall resistance
- Applications in dc and ac electrical metrology
Discovery of the Quantum Hall Effect

K. Von Klitzing discovers the quantum Hall effect in on 5 February 1980 in Grenoble

Classical Hall effect (1879)

Lorentz Force

\[ \vec{F}_L = q(\vec{v} \times \vec{B}) \]
Classical Hall effect (2)

2D-system:

\[ U_H = \frac{B \cdot I}{n \cdot e} \]

Hall effect independent of geometrical dimensions!

\[ U_H = \frac{B \cdot I}{n_{3D} \cdot e \cdot d} \]

\( n_{3D} \): carrier concentration
Realisation of a 2D electron gas
GaAs Heterostructure

Layers grown by: Molecular Beam Epitaxy (MBE) or Metal Organic Chemical Beam Epitaxy (MOVCD)
Inversion Layer

- de Broglie wavelength: typically 100 nm (GaAs)
- Level spacing at 10 T:
  - Si: 7.9 meV
  - GaAs: 17 meV
  - Graphene: 120 meV
- Thermal energy:
  - $T = 4\, \text{K}$: $kT = 0.36\, \text{meV}$
Landau levels

cyclotron motion in a strong magnetic field

Classical:

$$\omega_c = \frac{v}{r_c} = \frac{e \cdot B_z}{m^*}$$

Quantum mechanics:
Energy values of closed orbits

$$E = E_0 + \hbar \omega_c \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, 3...$$

magnetic length:
(8 nm @ 10 T)

$$r_c = l \sqrt{2n+1}, \quad l = \sqrt{\frac{\hbar}{e \cdot B}}$$
Landau quantization (2)

Orbital degeneracy

\[ N = \frac{L \cdot w}{2\pi l^2} \]

Number of states in a Landau level

State density

\[ n_B = \frac{1}{2\pi l^2} = \frac{eB}{h} \]

Number of flux quanta within the area of the sample

Filling factor:

\[ i = \frac{n_s}{n_B} \]
Quantum Hall effect?

Hall voltage in a 2D-system:

\[ U_H = \frac{B \cdot I}{n_s \cdot e} \]

\[ R_H = \frac{B}{i \cdot n_B \cdot e} = \frac{h}{e^2 \cdot i}, \quad i = 1, 2, 3... \]

observed when \( i \) levels are fully occupied!
Disorder and scattering removes orbital degeneracy
Localized states do not carry current
Plateau forms when $E_F$ resides within the localized states
QHE in a real device

\[ V_H + V_C \]

\[ I \]
QHE in a real device (2)

No energy gaps for real devices with finite width
Edge state picture

Skipping Orbits
source and drain contact are connected by a common edge
one-dimensional edge channels carry the current

Landauer formalism
current = driving force of electronic transport
Büttiker formalism

Current in 1D channels:

\[ I = \frac{e}{h} \Delta \mu \]

\( \Delta \mu \): difference electrochemical potential

\( T \): transmission

\( R \): reflection coefficient

\( T_i = 1, \ R_i = 0 \)

\( \mu_1 = \mu_3 = \mu_4; \ \mu_2 = \mu_5 = \mu_6 \)

\[ R_H(i) = \frac{(\mu_5 - \mu_3)}{e} \frac{h}{i \cdot e(\mu_2 - \mu_1)} = \frac{h}{i \cdot e^2} \]

\[ R_{xx} = \frac{(\mu_4 - \mu_3)}{e} \frac{h}{i \cdot e(\mu_2 - \mu_1)} = 0 \]
QHE Model

- The QHE is a collective phenomenon, it can not be explained by a microscopic model.
- A vanishing longitudinal resistivity indicated the absence of backscattering.
- Under this condition, the QHE is the direct consequence of the transmission of one-dimensional channels.
QHE in Graphene

Graphene: 2D crystal of carbon atoms with charge carriers » massless relativistic particles

Geim & Novoselov
Nobel Prize in Physics (2010)
Unconventional quantum Hall effect

\[
\sigma_{xy} = \left( \frac{4e^2}{h} \right) (N + \frac{1}{2})
\]

Novoselov, et al, Nature 438, 2005
Interest for graphene

\[ \Delta E \text{ required for } 10^{-9}\text{-accuracy of } R_H \text{ at } T \]

Potential to develop a quantum Hall resistance standard
At \( T > 5 \text{ K} \) and \( B < 5 \text{ T} \)

\( \Rightarrow \) Cryogen-free and cheap
Fine structure constant

\[ R_K \equiv \frac{h}{e^2} = \frac{\mu_0 c}{2\alpha} \]

Test the validity of

\[ R_K = i \cdot R_H (i) \]

or: additional route to the determination of \( \alpha \) (independent of QED)

most accurate value for \( \alpha \):

measurement of \( a_e \) + theoretical expansion of \( a_e \) in a series expansion of \( \alpha \) (numerical computations: Kinoshita et al.)
Fine structure constant

\[ R_K \equiv \frac{\hbar}{e^2} = \frac{\mu_0 c}{2\alpha} \]

- **\( a_e \):** Anomalous magnetic moment of \( e \)
- **\( \Delta \mu_{Mu} \):** Ground state hyperfine splitting
- **\( \Gamma_{90} \):** Gyro-magnetic moment of the proton
- **\( h/m \):** Neutron diffraction; cold atoms..
- **\( R_K \):** QHE

\[ (\alpha^{-1} - 137.03) \times 10^5 \]

\( h/m \) (Cs) \quad \( h/m \) (Rb)

\( \Delta \nu_{\text{Mu}} \)

\( a_e \) (Wash) \quad \( a_e \) (Harv U) \quad \( \text{CODATA-14} \)

\( R_K \) (NPL) \quad \( R_K \) (LNE) \quad \( R_K \) (NMI) \quad \( R_K \) (NIST)

QED theory necessary

without QED
Metrological application

• Ideal systems: $T = 0 \ \text{K}, \ I = 0 \ \text{A}$

• No dissipation: $R_{xx} = 0$

$R_H(i) = \frac{h}{i \cdot e^2} = \frac{R_K}{i}$

$R_K$ is a universal quantity
Localization theory,
Edge state model

• Real experiment: $T > 0.3 \ \text{K}, \ I = 40 \ \mu\text{A}$
Non-ideal samples

• Dissipation: $R_{xx} > 0$

$R_H(i, R_{xx} \to 0) = \frac{h}{i \cdot e^2}$

Is $R_H(i)$ a universal quantity?
Independent of device material, mobility, carrier density, plateau index, contact properties.....?

Few quantitative theoretical models available
→ empirical approach,
→ precision measurements
QHE devices for metrology

Carrier concentration

\[ B(i) = n_s \cdot e \cdot R_H(i) \]

\[ 2 \times 10^{15} \text{ m}^{-2} < n_s < 7 \times 10^{15} \text{ m}^{-2} \]

\[ n_s > 7 \times 10^{15} \text{ m}^{-2} : \text{2nd subband fills up} \]

Mobility:

\[ \mu > 10 \text{ T}^{-1} \] to have clear separation of the LL up to plateau 4

Hall bar defined by photolithography and wet etching techniques

Alloyed AuGeNi contacts
QHE in a real device

\[ R_H = \frac{h}{i \cdot e^2} \]
Temperature dependence

- Thermal activation:
  \( 1 \text{ K} \leq T \leq 10 \text{ K} \)
electrons thermally activated to the nearest extended states

\[
\sigma_{xx}(T) = \sigma_{xx}^0 \cdot e^{-\Delta/kT}
\]
\[
\Delta = E_F - E_{LL}
\]

\[
\delta\sigma_{xy}(T) = \sigma_{xy}(T) - \frac{ie^2}{h}
\]

\[
\delta\rho_{xy}(T) = s \rho_{xx}(T)
\]

Cage et al. 1984:
1.2 K < \( T < 4.2 \) K,
2 GaAs samples, \( i = 4 \)
-0.01 < \( s < -0.51 \)
Current dependence

GaAs
\( T = 0.3 \text{ K} \)
\( i = 2 \)
\( W = 400 \text{ µm} \)

Breakdown
Measurement technique

- To make use of the QHE for metrological applications, a measurement technique, capable of transferring the QHR to room temperature resistance standards must be available.

- Most accurate resistance bridge technique:

  → Cryogenic Current Comparator (CCC)
The cryogenic current comparator (CCC): Principles

**Meissner effect:**

\[ \text{SQUID} \propto n_1 I_1 - n_2 I_2 \]
The CCC bridge:

\[ N_p \cdot I_P = N_S \cdot I_S \cdot (1+d) \]

with \( d = \frac{N_t}{N_s} \cdot \frac{R_L}{R_L + R_H} \)

Detector:

\[ U_m = R_S \cdot I_S - R_P \cdot I_P \]

\[ \frac{R_P}{R_S} = \frac{N_P}{N_S} \cdot \frac{1}{1+d} \cdot \frac{1}{1+\frac{U_m}{U}} \]
Ratio accuracy:

\[ W_1 = N(1 + \delta w_1) \]
\[ W_2 = N(1 + \delta w_2) \]
\[ U_{\text{SQUID}} \propto (\delta w_1 - \delta w_2) \]

Windings in a binary series:
1, 1, 2, 4, 8, 10, 16, 32, 32, 64, 100, 128, 256, 512, 1000, 1097, 2065, 4130
Ratio accuracy

Accuracy:

- $0.3 \times 10^{-9}$ at $N = 2$
- $< 0.1 \times 10^{-9}$ at $N > 10$
Performance

SQUID noise
$7 \times 10^{-5} \frac{\phi_0}{\sqrt{\text{Hz}}}$

Transfer function
$\frac{\Delta I_{CC}}{\Delta \phi_s} \approx 4 \mu\text{A} \cdot \text{turn}/\phi_0$

Thermal noise
$V_{n-th} = \sqrt{4k_B T \cdot R \cdot B}$

$R_H(2) : 100 \ \Omega$
$N_P = 2065, N_S = 16, I_P = 50 \ \mu\text{A}$
$V_{n-rms} : \ 7 \text{ nV}/\sqrt{\text{Hz}}$
$u_A : \ 1 \text{ n}\Omega/\Omega \text{ in 2 min}$
Universality of the quantum Hall effect

- Width dependence
- Contact resistance
- Device mobility
- Plateau index
- Device material: MOSFET-GaAs - Graphene
Geometry of the QHE device

Some theories predict a **width dependence** of the QHR

No size effect observed within the measurement uncertainty

![Geometry Diagram]

![Graph of ΔR_H/R_H vs w^2]

Jeanneret et al., 95

Varenna 2016 / El. Standards II 37
Effect of the contact resistance $R_c$

M. Büttiker, 1992: “...It is likely, therefore, that in the future, contacts will play an essential role in assessing the accuracy of the QHE.”

On a perfectly quantised plateau:

$$R_c(P_1) = \frac{\Delta V_{P_1P_2}}{I}$$

**Real sample**

- $R_c > 0$
  - $\rightarrow$ Transmission $\neq 1$
  - $\rightarrow$ Reflection $\neq 0$

- Bad contacts
  - $\rightarrow$ electron gas depletion in the contact region
  - $\rightarrow$ non-ohmic behaviour of metal semiconductor interface
Contact resistance (2)

\[ R_H = \frac{h}{e^2} \left( \frac{1}{v_g} - \frac{1}{v} \right) \]

\( v_g \): depleted region in the contact arm

\[ h/2e^2 \quad h/6e^2 \quad h/12e^2 \]
Contact resistance (3)

Deviation of $R_H$ related to finite $V_{xx}$
Scattering parameters

$R_H$ independent of the device mobility or the fabrication process to 2 parts in $10^{-10}$

Electron mobility: measure of the electron velocity

Jeckelmann et al. 97

$\Delta R_H/R_H$ (n$\Omega$/$\Omega$)

Mobility (T$^{-1}$)
Step ratio measurements

$R_H$ independent of the plateau index to 3 parts in $10^{-10}$

$$\frac{i \cdot R_H(i)}{2 \cdot R_H(2)} = 1 - (1.2 \pm 2.9) \times 10^{-10}$$

$i = 1, 3, 4, 6, 8$

Jeckelmann et al. 97
QHR comparison GaAs - MOSFET

\[ \frac{\Delta R_H (\text{MOSFET} - \text{GaAs})}{R_H} \leq 2.3 \times 10^{-10} \]

Jeckelmann et al., METAS, 1996
QHR comparison GaAs - Graphene

\[ \Delta_{\text{GaAs-graphene}} = (-4.7 \pm 8.6) \times 10^{-11} \]

High precision measurements in epitaxial graphene on SiC at NPL
QHR comparison GaAs – Graphene (2)

F. Lafont et al. LNE, 2014: graphene grown by CVD on SiC

\[ \Delta_{GaAs-graphene} = (-0.9 \pm 8.2) \times 10^{-11} \]
Universality: Summary

The quantum Hall resistance is a universal quantity independent of:

- Device width
- Device material: MOSFET- GaAs - Graphene
- Device mobility
- Plateau index

.....to a level of \(< 10^{-10}\)

\[ R_H(i, R_{xx} \to 0) = \frac{h}{i \cdot e^2} \]

**CCEM Technical Guidelines:**
International QHR Key-comparison

Key Comparison
BIPM.EM-K12
Application: DC Resistance Standard (data METAS)

- Deviation from fit < 2 nΩ/Ω over a period of 10 years
QHE arrays

Connection of several Hall bars
(Delahaye 1993)

\[ R_{AB} = 2R_H(1 + \delta_c) \]

**single connection**

\[ \delta_c \approx \frac{R_c}{R_H} \]

**double connection**

\[ \delta_c \approx \left( \frac{R_c}{R_H} \right)^2 \leq 10^{-10} \]
QHE array (LNE)

129 \( \Omega \) @ \( i = 2 \)
AC-QHE: Applications

- SI realisation of the Farad: Calculable capacitor
  - complicated experiment
- Representation of the Farad: DC QHE
- **New route**: AC measurements of the QHR
First AC measurements of the QHE

- Narrow bumpy “plateau “ (PTB, NPL, NRC, BIPM)
- Frequency dependence:
  \[ \frac{\Delta R_H(i, \omega)}{R_H} = \alpha \omega \]
  \( \alpha = 1 \) to 5 \( 10^{-7} \)/ kHz

Measurement problems: AC Losses

Ratio Bridge: Principle

\[ i_D = i_T + i_B + i_Q = 0 \]

\[ \Rightarrow \frac{1}{Z_T} - \frac{1 + \alpha}{Z_B} - j \omega C \beta = 0 \]

\[ \frac{R_B}{R_T} = 1 + \alpha \]

\[ R_H(\omega, I) = R_c(\omega) \cdot (1 + \alpha(\omega, I)) \]
Calculable Resistor: Quadrifilar Resistor (QR)

Transmission line equations

\[ R_{//} = f(R, L, C, G, \omega) \]

\[ R_c(\nu) = R_c^{dc}(1 + \delta \cdot \nu^2) \]

\[ \delta = 0.0129 \times 10^{-6}/k\text{Hz}^2 \]
AC measurements of the QHE
(with grounded back gate)

Linear frequency dependence of the Hall resistance observed

[Graph showing capacitively losses scale with the surface]
Model for ac Losses

\[ Z_H = \frac{V_H}{i} = \frac{V_H}{i_H - i_l} \approx R_H \left( 1 + \frac{R_H}{V_H} i_l \right) = R_H (1 + \Delta) \]

Overney et al., 2003
Adjusting capacitive losses
(2 back gates)
Double shielding technique

Meet the defining condition: ALL currents which have passed the Hall-potential line are collected and measured.

Adjust the high-shield potential $sU$ so that $dR_H/dI = 0$.

QHR as impedance standard

\[ \frac{\Delta R_H}{R_H} / 10^{-9} \]

\[ f / \text{kHz} \]

\[ \sigma = 1.4 \cdot 10^{-9} \text{ kHz}^{-1} \]

\[ \Rightarrow \text{capacitive effects} < 1.4 \cdot 10^{-9} \text{ kHz}^{-1} \text{ for different devices} \]

\[ \Rightarrow \text{better than artefacts} \]

Realization of the Farad

- relative uncertainty of 10 pF: \(4.7 \cdot 10^{-9}\) \((k = 1)\)
  (cryogenic quantum effect without 'calculable' artefacts)
- quantum standard of capacitance, analogous to \(R_{DC}\)

Conclusions

- $R_H$ is a universal quantity
- QHR improved electrical calibrations in National Metrology Institutes considerably
- QHR plays an important role in the determination of the fine structure constant
- QHR can be used as quantum standard for impedance
Thank you very much for your attention