Tilted and standard ring solitons: from plasma to shallow water

Tilted and standard ring solitons in shallow water

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In many nonlinear systems, ranging from plasma to the ocean, the interplay between the nonlinearity and dispersion leads to the formation of a very robust coherent and localized structures, the solitons.

In cylindrical symmetry, the theory of the large amplitude waves predicts the existence of nonlinear ring waves in the form of multi-solitons, whose spatiotemporal evolution is governed by the concentric (or cylindrical) Korteweg-de Vries equation (cKdVE).

Here, we solve both analytically and numerically the cKdVE. The analytical multi-soliton ring waves are associated with the “tilted” boundary conditions (“tilted solitons”) i.e., for large values of the spatial coordinate, the solution approaches an oblique asymptote.

Furthermore, the ring solitons obtained from numerical solutions are associated with “standard” boundary conditions for a bright multi-soliton, i.e., the solutions tend to zero asymptotically. Although the maximum amplitudes of both analytical and numerical solutions decrease as the timelike variable increases, they exhibit a different scaling law.

Remarkably, it is shown that the numerical cylindrical multi-soliton solutions exhibit a behaviour that is very similar to the planar KdV multi-soliton.
These pictures have been taken from Google Images
A number of different modes have been investigated in both pure and dusty plasmas, such as electron acoustic, ion acoustic, dust electron acoustic, dust ion acoustic (DIA), dust acoustic waves, etc. For instance, DIA waves in an unmagnetized dusty plasma are typically described by assuming: inertial ions, Boltzmann-distributed electrons, and negatively charged immobile dust particles [1]. The corresponding mathematical model is given by the following equations

\[
\frac{\partial n_i}{\partial t} + \frac{1}{r^\nu} \frac{\partial}{\partial r} \left( r^\nu n_i u_i \right) = 0 \quad \frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial r} = - \frac{\partial \phi}{\partial r} \\
\frac{1}{r^\nu} \frac{\partial}{\partial r} \left( r^\nu \frac{\partial \phi}{\partial r} \right) = \mu \exp(\phi) - n_i + (1 - \mu)
\]

\[
\mu = n_e / n_i, \quad \nu = 0 \text{ (planar geometry), } \nu = 1 \text{ (cylindrical geometry), } \nu = 2 \text{ (spherical geometries)}
\]

\[
\xi = -\epsilon^{1/2} (r + \nu_0 t) \quad \tau = \epsilon^{3/2} t + \text{ Reductive Perturbation Method}
\]

\[
\phi^{(1)} \text{ is the electrostatic wave potential.}
\]

Schematic Diagram

Water Surface Waves

Picture has been taken from Google Images
Water is assumed to be: incompressible, irrotational, inviscid fluid, constant density, zero surface tension, constant depth, constant atmospheric pressure \([2 - 4]\)

**The governing equations for cylindrical geometry:**

\[
\begin{align*}
\frac{\partial u}{\partial t} + \epsilon (u \cdot \nabla) u - \frac{\epsilon v^2}{r} &= -\frac{\partial p}{\partial r}, \\
\frac{\partial v}{\partial t} + \epsilon (u \cdot \nabla) v + \frac{\epsilon uv}{r} &= -\frac{1}{r} \frac{\partial p}{\partial \theta}, \\
\frac{\partial w}{\partial t} + \epsilon (u \cdot \nabla) w &= -\frac{\partial p}{\partial z}, \\
\frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} &= 0,
\end{align*}
\]

\[
p = \eta = \text{constant and } w = \frac{\partial \eta}{\partial t} + \epsilon \left( u \frac{\partial \eta}{\partial r} + \frac{v}{r} \frac{\partial \eta}{\partial \theta} \right) \text{ on } z = 1 + \epsilon \eta
\]

\[\epsilon = \frac{a}{h_0}, \text{ measures the nonlinearity } \quad \delta = \frac{h_0}{\lambda}, \text{ estimates the shallowness of the water}
\]

After performing a Taylor expansion in the small parameter \(\Delta = \epsilon^4 / \delta^2\) of \(H, P, U\) and \(W\), we obtain equation for \(H_0\) (i.e., the surface elevation at the leading order):

\[
\frac{\partial H_0}{\partial s} + \frac{3}{2} \frac{H_0}{\partial \zeta} + \frac{1}{6} \frac{\partial^3 H_0}{\partial \zeta^3} + \frac{H_0}{2s} = 0
\]

\[
\zeta(R,T) = R - T, \quad s(R,T) = T \quad v(R,T) = H_0(\zeta(R,T), s(R,T))
\]

\[
\frac{\partial v}{\partial T} + \frac{\partial v}{\partial R} + \frac{3}{2} v \frac{\partial v}{\partial R} + \frac{1}{6} \frac{\partial^3 v}{\partial R^3} + \frac{v}{2T} = 0
\]

\[\text{cKdVE for cylindrical symmetry}\]

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Analytical Solutions

- **Single Soliton Solution [3-7]:**
  \[ H_0(\zeta, s) = \frac{2}{3s} \left\{ \frac{\zeta}{2} + \frac{3}{2} u_m \text{sech}^2 \left( \sqrt{\frac{3u_m}{4s}} (\zeta + u_m) \right) \right\} \]

- **Two Soliton Solution [3,4]:**
  \[ H_0(\zeta, s) = \frac{2}{3s} \left( \frac{\zeta}{2} + \frac{3}{2} (u_{m2} - u_{m1}) \frac{P' + Q'}{Sr^2} \right) \]

- **Three Soliton Solution [3,4]:**
  \[ H_0(\zeta, s) = \frac{2}{3s} \left( \frac{\zeta}{2} + \frac{3}{2} \left\{ P' - \frac{(u_{m2} - u_{m3}) W'}{(u_{m1} - u_{m2} - u_{m3} - u_{m1})^2} \right\} \right) \]

\[ P' = u_{m1} \text{sech}^2 \left( \sqrt{\frac{3u_{m1}}{4s}} (\zeta + u_{m1}) \right) \]
\[ R' = u_{m3} \text{sech}^2 \left( \sqrt{\frac{3u_{m3}}{4s}} (\zeta + u_{m3}) \right) \]
\[ S' = \sqrt{u_{m1}} \tanh \left( \sqrt{\frac{3u_{m1}}{4s}} (\zeta + u_{m1}) \right) - \sqrt{u_{m2}} \coth \left( \sqrt{\frac{3u_{m2}}{4s}} (\zeta + u_{m2}) \right) \]
\[ T' = \sqrt{u_{m3}} \tanh \left( \sqrt{\frac{3u_{m3}}{4s}} (\zeta + u_{m3}) \right) - \sqrt{u_{m1}} \tanh \left( \sqrt{\frac{3u_{m1}}{4s}} (\zeta + u_{m1}) \right) \]
\[ W' = \frac{(u_{m2} - u_{m1})(P' + Q')}{S'^2} + \frac{(u_{m3} - u_{m1})(R' - P')}{T'^2} \]
\[ Q' = u_{m2} \text{cosech}^2 \left( \sqrt{\frac{3u_{m2}}{4s}} (\zeta + u_{m2}) \right) \]

To find the solution of cKdVE from KdVE

\[
\frac{\partial u}{\partial \tau} + \frac{3}{2} u \frac{\partial u}{\partial \xi} + \frac{1}{6} \frac{\partial^3 u}{\partial \xi^3} = 0
\]

\[
u(\xi, \tau) = u_m \text{sech}^2 \left[ \sqrt{3u_m/4} (\xi - V_0 \tau) \right] \quad \text{where } V_0 = u_m/2
\]

\[
H_0(\zeta, s) = s^{-1} \left[ u \left( \xi = s^{-1/2} \zeta, \tau = -2s^{-1/2} \right) + \zeta/3 \right]
\]

\[
v(R, T) = \frac{1}{T} \left[ u \left( \frac{R-T}{T^{1/2}}, -\frac{2}{T^{1/2}} \right) + \frac{R-T}{3} \right]
\]

\[
v(X, Y, T) = \frac{1}{T} \left[ u \left( \frac{\sqrt{X^2+Y^2}-T}{T^{1/2}}, -\frac{2}{T^{1/2}} \right) + \frac{\sqrt{X^2+Y^2}-T}{3} \right]
\]

\[
R = \sqrt{X^2+Y^2}
\]

Plots of Analytical Solutions

- Single Soliton: 1D, 3D
Plots of Analytical Solutions

Two Soliton: 1D, 3D
Plots of Analytical Solutions

Three Soliton: 1D, 3D
Numerical Solutions

- **Initial conditions:**
  - Single Soliton:
  - Two Soliton:
  - Three Soliton:

- **Boundary conditions:**

\[
v(R, T_{\text{min}}) = R^n u_m \text{sech}^2 \left( \sqrt{\frac{3u_m}{4}} (R - R_0) \right)
\]

\[
v(R, T_{\text{min}}) = (u_{m2} - u_{m1}) \frac{P + Q}{S^2}
\]

\[
v(R, T_{\text{min}}) = P - \frac{u_{m2} - u_{m3}}{\left( \frac{u_{m1} - u_{m2}}{S} - \frac{u_{m3} - u_{m1}}{L} \right)^2} \left[ \frac{(u_{m2} - u_{m1})(P + Q)}{S^2} + \frac{(u_{m3} - u_{m1})(F - P)}{L^2} \right]
\]

\[
P = u_{m1} \text{sech}^2 \left( \sqrt{\frac{3u_{m1}}{4}} (R - R_1) \right)
\]

\[
Q = u_{m2} \text{cosech}^2 \left( \sqrt{\frac{3u_{m2}}{4}} (R - R_2) \right)
\]

\[
F = u_{m3} \text{sech}^2 \left( \sqrt{\frac{3u_{m3}}{4}} (R - R_3) \right)
\]

\[
S = \sqrt{u_{m1}} \tanh \left( \sqrt{\frac{3u_{m1}}{4}} (R - R_1) \right) - \sqrt{u_{m2}} \coth \left( \sqrt{\frac{3u_{m2}}{4}} (R - R_2) \right)
\]

\[
L = \sqrt{u_{m3}} \tanh \left( \sqrt{\frac{3u_{m3}}{4}} (R - R_3) \right) - \sqrt{u_{m1}} \tanh \left( \sqrt{\frac{3u_{m1}}{4}} (R - R_1) \right)
\]

\[
\lim_{R \to 0} v(R, T) = \lim_{R \to \infty} v(R, T)
\]
Plots of Numerical Solutions

👀 Single Soliton Case: 1D, 3D
Two Soliton Case: 1D, 3D
Plots of Numerical Solutions

- Three Soliton Case: 1D, 3D
Break up into Multiplet of Ring Solitons

\[ v(R, T) \]

- \( T = 5 \)
- \( T = 5.5 \)
- \( T = 6 \)
- \( T = 9 \)
- \( T = 15 \)
- \( T = 20.6 \)
Fit Results

Figures a and b: Time evolution of the maximum amplitude (Fig. a) and width (Fig. b) calculated by the solution $v(R, T)$. The data points are the results of the numerical calculations. Solid lines represent the fit of the amplitude data (Fig. a) by the power law $a + bT^{-\alpha}$ and of the width (Fig. b) by the power law $p + qT^\beta$. 

Single Soliton Case
Conclusions & Remarks

1. The spatiotemporal analytical tilted multi-soliton-like structures:
   - do not exhibit overlapping of the individual pulses, because they are propagating with the same speed (this means that there is no a relationship between the pulse height and its speed);
   - are not accompanied by water wakes behind or ripples at the front.
   - The physiological flattening of the analytical multi-soliton structures is associated with both pulse amplitude decay and width increase, that scale as $1/T$ and $\sqrt{T}$, respectively.

2. The numerical multi-soliton packets exhibit an internal nonlinear dynamics, very similar to that one existing in the planar KdVE:
   - the internal dynamics is governed by a monotonic relationship between the soliton-like pulse height and its speed.
   - the nonlinear character makes manifest through the overcoming of a train of solitons by a pulse of higher amplitude, which moves at faster velocity.
   - soliton overlapping and the pulse radiation i.e., the formation of water wakes behind the pulse packets, depends on the initial conditions of the train of solitons.
   - the physiological flattening of the numerical multi-soliton structures involves the decay of the maximum amplitude of the pulse and the increase of its width, that scale approximately as $1/\sqrt{T}$ and $T$, respectively.
Thank you very much for your kind attention!

WE WELCOME YOUR QUESTIONS, SUGGESTIONS, COMMENTS!