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The role of mathematics for physics teaching and understanding

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Summary. — That mathematics is the "language of physics" implies that both areas are deeply interconnected, such that often no separation between "pure" mathematics and "pure" physics is possible. To clarify their interplay a technical and a structural role of mathematics can be distinguished. A thorough understanding of this twofold role in physics is also important for shaping physics education especially with respect to teaching the nature of physics. Herewith the teachers and their pedagogical content knowledge play an important role. Therefore we develop a model of PCK concerning the interplay of mathematics and physics in order to provide a theoretical framework for the views and teaching strategies of teachers. In an exploratory study four teachers from Germany and four teachers from Israel have been interviewed concerning their views and its transfer to teaching physics. Here we describe the results from Germany. Besides general views and knowledge held by all or nearly all teachers we also observe specific individual focus depending on the teachers' background and experiences. The results fit well into the derived model of PCK.

1. – Introduction

The description of physical processes by mathematical means is one of the most characteristic traits and most powerful tools of physics research, and so it is undisputed that mathematics is the language of physics (see, *e.g.*, Dirac 1939, Wigner 1962, Bochner 1963, Hestenes and Sobczyk 1987). Both are closely interrelated and even more: often mathematical structures are inherent in physical concepts. Therefore the nature of physics as an empirical science where mathematics and physics are closely intertwined should be a central aspect of physics education. As its goal consists in teaching the principles, concepts and methods of physics, in short: the nature of physics the use of mathematical elements should be an intrinsic feature of teaching, adapted to the respective school level. However, understanding the meaning of different mathematical tools and their interrelation with the physical description of the world seems to be one of the most difficult steps in physics learning. In investigating this interrelation from an educational perspective, many questions arise. The overarching question is: Can the domain of mathematical structures improve the understanding of physical concepts and if so, in which way? How can the interplay of mathematics and physics be taught appropriately? In physics lessons teachers, their views and their competences, in short: their pedagogical content knowledge play a central role for the successful learning of students. Up to now only little is known about the knowledge and views of teachers in the important area of shaping the interplay of mathematics and physics. Therefore this central point of teaching physics should come into focus of physics education research.

2. – Theoretical framework

The study of physics at university incorporates mathematics as a prerequisite which often represents a difficulty for students in problem solving and applying mathematics appropriately to physics problems (*e.g.*, Tuminaro and Redish 2007, Sherin 2001). One could ask oneself where these difficulties are rooted. Generally, *e.g.* in Germany the curricula and exams in high school regard the use of mathematics in physics as important and also as preparing for physics study at university (KMK 2006). However, in junior high school (grades 6 to 10) physics is generally taught in a more qualitative approach (KMK 2004). Only in few instances mathematics really is applied in a deeper way. Nevertheless as in these middle grades the first steps towards the later and more extensive use of mathematics are done, it is important to analyse which aspects of mathematics in physics are central and how they could be taught. The proper shaping of the interplay of mathematics and physics in physics education is important for giving the students an insight not only in physics but also into the nature of physics, (Hestenes 2013). On the whole we suppose no sharp distinction between the mathematical and physical models (Greca and Moreira 2002) but more an inseparable intertwinning of both science areas.

2˙1. *Role of mathematics in physics and in physics education.* – The role of mathematics in physics covers different aspects: Pietrocola (2008) distinguishes a technical and a structural role. Since here not always a sharp border can be defined, we differentiate in addition the modelling aspect, where the results of physical experiments and observations are ordered with help of an abstract symbolic system, and the communicative aspect, *i.e.* using and setting into relation to each other different representations (Krey 2012). In general, mastering the purely technical role is not sufficient for being successful in physics (Bing and Redish 2009). Besides the technical aspects of mathematical syntax, the calculating and solving (differential) equations, the semantics, the physical meaning of mathematical structures, play an important role (Sherin 2001). The structural role implies that physics inherits the formal operations and definitions of mathematical objects, thus allowing for formal derivations of physical laws not thinkable of before. In addition mathematics might order the physical phenomena according to underlying patterns (modelling and hinting at analogies) and in a further extension it enriches physical thought by the physical meanings of mathematical operations (operations, limiting cases, functions, . . .).

In order to mirror these aspects in physics education not only the technical but above all the structural role must be taught intentionally. Students have to learn how to use mathematics for structured thinking about physical processes and how to interpret mathematical structures in physical terms. Different mathematical elements or representations such as geometrical objects, diagrams, algebraic expressions and verbal explanations have their specific roles for supporting understanding by describing exactly, quantifying and visualizing physical processes (Pospiech 2007). Their use requires mathematical abilities on very different levels: it starts with recognizing and verbalizing functional dependencies/*e.g.* the more .. the more) and then goes to the quantitative description with diagrams or algebraic expressions including interpreting diagrams and formula in terms of physical concepts. The next more complex step consists of applying techniques of modelling and idealization. A final, quite advanced step would be the mathematical formulation of basic physical principles in the framework of theories which is for the most part beyond school physics (*e.g.*, Noether theorems).

Especially for students in junior high school the first steps towards the mathematical description of physics are crucial (Pospiech 2007). The path from a phenomenological level up to a more abstract level where mathematical reasoning concerning physical laws and processes can take place has to be shaped carefully by treating both the technical and the structural aspect. In order to grasp possible patterns in this process of mathematisation a model has been developed focussing on the structural aspect and leaving open the possibility of very different strategies and patterns for deriving mathematical descriptions, depending on the mathematical tools available (Uhden *et al.* 2012). This model can serve for classifying difficulties of students in solving problems (Uhden 2012). In addition it can serve as an analytic tool for the teaching process in that it shows which mathematisation steps are necessary and indicates how difficult they might be (Müller 2012). Therefore it gives hints for students' ideas as well as for instructional strategies.

2˙2. *Role of pedagogical content knowledge of teachers*. – Teachers and their professional knowledge play a central role for the learning of students (Krauss *et al.* 2008, Riese 2010, Shulman 1986). Shulman (1986) identified the so called pedagogical content knowledge (PCK) as "amalgam of content knowledge and teaching abilities". Some studies (*e.g.*, COACTIV) show a positive effect of good PCK for learning of students and cognitive activation (Kunter *et al.* 2011). There are different models of PCK (Gramzow *et al.* 2013). However, most models agree that, *e.g.*, knowledge of explanations and representations and knowledge of students thinking are important. We are using the model of Magnusson *et al.* (1999). It unfolds the PCK in five dimensions which are interconnected: Orientations Toward Teaching, Knowledge of Science Curriculum, Knowledge of Students' Understanding of Science, Knowledge of Assessment in Science and Knowledge of Instructional Strategies.

By analogy the consistent teaching of the complex interplay of mathematics and physics requires that the teachers are aware of basic metacognitive ideas concerning the general, structural role of mathematics in physics and use them to shape the interplay regarding the students ideas and abilities on the different educational levels. For this purpose the teachers should develop an appropriate PCK adapted to the role of mathematics in physics education.

3. – Research questions

In general the topic of the role of mathematics in physics education and the knowledge, the teaching strategies and views of teachers about it are quite unresearched until now. Taking into account the central role of mathematics, especially its structural aspects, this research gap is astonishing. In order to get insight about this aspect an exploratory study with experienced or expert teachers (which might be distinguished) seems to be an appropriate way to get an idea of what is possible and how a PCK about "role of mathematics in physics and in physics education" could look like. In this first preliminary study also the occurrence of possible strategies in teaching should be explored. The questions treated in this contribution are:

- What do experienced teachers think about the role of mathematics in physics?
- Which strategies do they use to convey the structural role of mathematics in physics?

4. – Methods and study

In order to answer the research questions in a first step we derive a specific model for the PCK of teachers concerning the interplay of mathematics and physics. This model contains the aspects described above. From this an interview guideline was developed for exploring the views of teachers in a first study with four teachers from Germany. The interviews with teachers of different background should also help to test the PCK-model and identify different characteristics of it among the teachers.

4·1. *Proposal for PCK of teachers with respect to mathematics in physics education.* – In developing the PCK-model we are drawing on the distinction of technical and structural role of mathematics for physics as described above. In order to unfold the components of PCK we remark that mathematics provides tools for representing and quantifying physical processes and quantities. Its structures have to be interpreted physically and can help in recognizing analogies. The intertwinning becomes obvious in physical concepts which cannot even be formulated without mathematics, as, *e.g.*, the velocity or acceleration. We expect teachers to be aware of all these theoretical considerations. In addition their professional knowledge should include the implications for enacting formal procedures and algorithms in class, *e.g.*, by fostering the interpretation of symbols, the meaning of functional dependencies and so on. In this field of the interplay between physics and mathematics the teachers should be able to move consciously and well reflected on their own views, their students' views and the objectively given interrelations. Basis for all this is the general orientation towards teaching and their explicit and implicit goals of physics teaching.

In the following we describe the different components of the PCK in accordance to the Magnusson model (1999) and give a detailed description with respect to our topic:

- Orientations Toward Teaching Physics

General aspects: The teachers should reflect on the impact of the broad philosophical or historical discussion about the role of mathematics in physics on their own views and teaching.

Metacognitive ideas about the interplay of mathematics and physics: The teachers should consider in which way the mathematisation can contribute to insight of their students into the nature of physics and into the physical method. They consider the role of deductive reasoning in physics teaching with help of mathematical techniques. They reflect on the relation of experiment and use of mathematics in their teaching.

- Knowledge of Physics Curriculum with Respect to Use of Mathematics

Knowledge of goals and objectives: Teachers can reflect on goals of the use of mathematics as expressed in the official curriculum and set these considerations into relation to their own views. They relate the curricula across the grades and are aware of the interplay of physics and mathematics curricula.

Knowledge of specific curricular program: Teachers should know concrete examples where they can apply mathematics in different degrees of complexity. They are aware of specific chances or difficulties, inherent in the curriculum.

- Knowledge of Students' Ideas

Knowledge of conditions of learning: Teachers know which mathematical and physical knowledge the students have to integrate. They are aware of general learning obstacles. Optimally they also know about the concrete mathematics prerequisites the students bring into their class.

Knowledge of specific aspects: Teachers can differentiate between purely technical and more structural problems of the students. a) Mathematics as a problem: Teachers know the most difficult points in the use of mathematics in physics, seen from the students' perspective. b) Mathematics as a useful help: Teachers know at which point the use of mathematics can support the students or motivate them towards the use of mathematics in physics.

- Knowledge of Instructional Strategies concerning the technical role of mathematics

Mathematics as a tool for communication (representation): Teachers know which mathematical representations they can use in teaching physics and how to integrate diagrams, algebraic and verbal representation appropriately.

Mathematics as a formal tool: Teachers are aware of critical points w.r.t. the technical role of mathematization. They know which kind of instruction, activities and exercises can be used and where to set their emphasis in order to ensure mastering technical aspects.

- Knowledge of Instructional Strategies concerning the structural role of mathematics

Mathematics as supporting physics understanding: Teachers know how to increase physics understanding by using appropriate mathematisation on the structural side. They foster basic understanding by structuring physics.

Knowledge of strategies shaping the interplay: Teachers know with which examples and methods they can give the students insight into the nature of physics and the physical method. a) Teachers use derivations and deductions in order to clarify relations and concepts. b) Teachers use experiments and relate them to mathematical descriptions of physical laws.

4·2. *Design of interviews and sample.* – The interviews covered the components of the PCK described above. The focus was the view of the teachers about the role of mathematics for physics teaching, the relevance for students learning and students' difficulties as well as examples of instructional strategies. The interviews were semi-structured and were conducted according to a guideline. The questions were very broad in order to explore the field. They were given to the teachers about one week in advance so that the teachers had the opportunity of preparing and making their view clear.

In Germany four teachers have been interviewed, two male and two female. All have studied mathematics as second subject in the same depth as physics and have more than 20 years of teaching experience. They are allowed to teach mathematics and physics on all levels from grade 5 to grade 12, the final grade before school-leaving exam. One teacher (T1) "Master Teacher") was involved in developing the current physics curriculum; one teacher (T2) is the head teacher for mathematics and physics in his school, implying above average qualification. One teacher (T3) has a focus on mathematics teaching, including advanced high school courses, nevertheless regularly teaching physics classes, mainly in junior high school up to grade 10. One teacher (T4) has a strong focus on teaching physics for younger students in the grades 6 to 10, but is also teaching mathematics.

Teacher T1 had a very broad preparation with refined arguments, teacher T2 did not prepare; the others were in between. The interviewer knew all of the teachers personally so that the atmosphere of the interviews were open and with confidence. The teachers volunteered for the interview because they are strongly convinced that mathematics is necessary for physics teaching, even in junior high school starting from grade 6. Two of the interviews were done by telephone. All lasted between 45 minutes to 60 minutes. The interviews have been audio-taped and transcribed.

4·3. *Method of data evaluation.* – Starting from the PCK-model categories were defined deductively. In the second step these categories were refined inductively from the material. The categories were validated in collegial discourse. Then the interviews were coded and discussed within the research group.

5. – Results of interviews with teachers in Germany

5·1. *Characterization of teachers.* – As the teachers agreed voluntarily to the interviews it was to be expected that they all gave the interplay of mathematics and physics in physics lessons a big relevance. However, the teachers showed individual key aspects in their views and own characteristics in their PCK as is seen in the distribution of the percentages of their statements in the different categories corresponding largely to the components of the PCK (see fig. 1).

With respect to the first research question, which refers to the component *Orientation towards Teaching Physics* with the subcategories *General aspects* and *Metacognitive ideas*

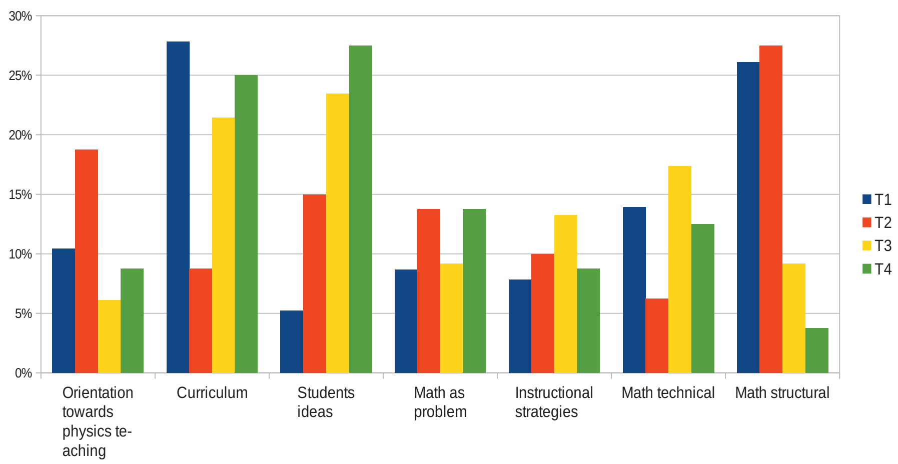

Fig. 1. – The percentages of the respective statements of the teachers are given. It is seen that there are clear differences between the teachers, especially concerning the students ideas and the awareness of the structural role of mathematics in teaching.

about the interplay of mathematics and physics it was said: "*that it is very important this relationship between mathematics and physics because in certain points profound insights in physics can only be reached with mathematics*" (T2) or: "*Without mathematics, physics is not possible*". Or that the "*mathematical methods are an important contribution to develop an understanding of the relationship between empiricism, i.e. experiment and theory*" (T1). For teacher T1 it is important that the mathematics always has a physical meaning and has to be interpreted: "*Moreover, the mathematical method is basis for the acquisition of knowledge in physics lessons.*" From a basic view the strongly mathematics oriented teacher T3 stresses that: "*The exactitude there is already important*".

On the whole the teachers had a quite homogeneous view on the interplay, but nevertheless they showed in their style different specifications in detail. Two teachers (T2 and T4) showed a strong concern that experiments are important for understanding and motivation and that the use of mathematics might not destroy this benefit. Teacher T3 had a very strong emphasis on the mathematical side with mainly technical concerns. Teacher T1 discussed at length the overall importance of mathematics along the physics curriculum.

Generally all the teachers have high knowledge of the curriculum, (component "*Knowledge of Physics Curriculum with Respect to Use of Mathematics*", see fig. 1). They can give concrete examples where the interplay of mathematics and physics plays a role, where it is demanded that they do it and where they do it voluntarily because they think it important or helpful. Often they even know precisely the curriculum of mathematics, partly because they are teaching this as well. It can be seen that the knowledge is especially precise, but not restricted to, on the topics or classes they teach more often or regularly. As a special point all the teachers state that the amount of formulas in physics curricula, hence the use of mathematics in physics teaching, has been largely reduced in the last ten years, especially in junior high school.

5·2. *Stressing the structural role of mathematics*. – As there are rich data and many information the teachers gave we cannot present all in this contribution. For answering the second research question we will focus on the teaching strategies and the combination of technical and structural role referring to the subcategory *Knowledge of strategies shaping the interplay* of the component "*Instructional strategies*". Here the number of statements differ very strongly between the teachers (see fig. 1). As some teachers value the experiments very highly this is an own aspect to be analysed. We observe different strategies of the teachers in order to use mathematical elements for supporting understanding. One is in the use of multiple representation, the other in changing back and forth ("interlacing") between the different roles of mathematics or between experiment and mathematical description.

Invoking different representations for stressing the structural role of mathematics in physics

We analyse descriptions of how the teachers employ different representations, mostly equations and diagrams where the diagrams often serve as a means for evaluating experiments and how they come, *e.g.*, from experiment to formula. This starts right with the beginning physics lessons as expressed by teacher T4: "*So in class 6 it starts, that drawing diagrams is transferred from the mathematics and then more and more practised*" with awareness of the complications: "*I, for example, already experienced that for the class 6 it is still a problem to draw a line graph*" (also see Mevarech and Kramarsky 1997). This is mostly combined with practical aspects: "*I'll introduce it [density] quite visual, by making them something to measure*". Even more pronounced is the visual aspect (also w.r.t. density) in the typical strategy of teacher T3 combining experiment and explicit calculation which uses all types of representations in a logical sequence: "*Then we have determined the mass of water for different volumes with an experiment. They have entered the values into a table and then they should draw the graph and just observe, where the proportionality could be. Then we have calculated the proportionality constant. We have also shown that there is quotient equality, so that we can call it proportionality*". Or in the explicit connection of diagram and proportionality: "*Consider: I have to double the heat for the same temperature change; so the diagram should look like that"*. Teacher T2 wants to show the importance of mathematics for practical aspects in the possibility to quantify physical quantities: "*Which wire do I have to buy if I want to build a suitable circuit? What diameter of the wire is necessary?*" or "*Now you can see on the basis of this equation what needs to be changed in this resonant circuit*". A more abstract explicitly metacognitive stance is taken by teacher T1: "*Mathematics is the basis for the evaluation of experiments, i.e. by collecting measured values in tables, estimating the uncertainty of measurement by calculation, etc.*". He stresses that mathematics can lead to new insights with help of diagrams and equations: "*Once I work with equations and diagrams in order to develop concepts and laws*" and in high school he uses even more complex tools: "*So I can formulate new equations or new contexts about these diagrams by understanding slope and area under the curve*".

All these statements show that for some teachers the practical aspect is in the foreground whereas other teachers also consciously try to convey the theoretical influence of mathematics by the possibility of deduction, often in combination with experiments. In both cases they use different representations and set them into relation to each other.

Shaping the interplay mathematics - physics by interlacing structural and technical aspects

Here we analyse strategies, sometimes also involving experiments. Sometimes a linear path from concrete to abstract can be observed, sometimes the way back is stressed.

For teacher T3 the whole path from the practical or experimental to the formula is very important: "*that [a student] describes the practical situation, can explain and that he can justify it mathematically in many places*". She explicitly refutes the purely technical procedure if students only mechanically try to solve problems: "*to take out a formulary, without thinking*" but the students should instead experience mathematics as supporting physics understanding "*They have really an understanding of why we do it this way*" and they should be able to explicitly combine the physical and the mathematical aspects: "*Density: grams per cm*^{3}*. Interpret this. Explain with the particle model*" or "*What is the impact of the constant factor?"*, requiring a clear connection between mathematical equations and physical concepts.

Teacher T4 sees that the mathematical form is not always absolutely necessary but that there are levels in between: "*This the more – the more - statements / you can do a lot with these statements. Actually you do not need the formula for it*", an intermediate semi-quantitative representation, so to say a "formula in words" between purely qualitative and purely formal and quantitative statements. This seems to be quite appropriate for her mostly younger students.

For teacher T1 mostly teaching higher grades the connection of calculations to applications is central: "*In the first instance we work with equations and diagrams to calculate physical quantities in the context of physical task*" and insists that the students have to learn that it is important to interpret any numerical result: "*But in the classroom, every lesson, if we calculate anything, I want to know: Is that a lot? Is that enough? Can it be? What does this tell us?*". This is the surface of the underlying structural role of mathematics in physics: "*Applying the mathematics so that the physical models really can be calculated*" and "*But I also think that these mathematical short forms are .. ultimately a means for recognizing analogous structures*". In a similar way teacher T2 is reasoning: "*I still put a lot of emphasis on the interpretation of equations*", but also with reference to practical applications or to deductive results: "*If you have an equation, then you can derive many things from the equation and then always imagine, what lies behind it practically*". Furthermore he very clearly addresses that this should be done in order to evade that "*The equations are always like a black box for the students*" and that there are examples where only mathematics can answer physical questions: "*The theory of relativity is indeed the place where you can clearly show the students that only with mathematics it was possible to get predictions regarding the time dilation, length contraction, everything else*".

So the main strategies again mirror the different experiences of the teachers. All insist that mathematical results have to be interpreted in physical terms and concepts or with respect to applications. Therefore the use of algebraic representation always is back and forth connected to experiments or to evaluation in terms of verbal representations. Teacher T2 stresses that there are areas of physics that can be treated fully only by mathematics (special relativity).

Setting individual emphasis throughout the curriculum

Each teacher has, depending on the grades and contents he or she is teaching mostly, developed his or her own specific main focus he lies emphasis on. It could be the practical side of mathematics, the specific difficulties of beginners or the working with models. This analysis will be given in a different paper.

6. – Conclusions and perspectives

In a model of PCK for teachers we elaborated on aspects relevant for teaching the interplay of mathematics and physics at school. The interviewed physics teachers were aware of its basics. They mostly took a practical stance, strongly influenced by the curriculum and dominated by realizing their strategies and taking into account typical problems and strategies of students, both from a technical and from a structural aspect. Based on these findings it would be interesting to look at actual lessons and to confirm the actual classroom practices of teachers and students with the goal of identifying successful patterns of teaching strategies. The focus should lie on junior high school because the basis is created here.

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