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A class of coupled nonlinear Schrödinger equations and their relations to the complex Hamilton system

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Summary. — In this paper, the relation between the coupled nonlinear Schrödinger equation and the complex Hamilton system are discussed by means of the third-order spectral problem. Then the complete integrability of the complex Hamilton system associated with the coupled nonlinear Schrödinger equation is discussed by using the symplectic structure and the complex representation of the Poisson bracket. So the solutions to the coupled nonlinear Schrödinger equation are derived.

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1. – Introduction

The relation between the finite-dimensional complete integrable Hamilton and the soliton equation has been an important question for study [1]. Though the complex complete integrable Hamilton system has been discussed, the complex complete integrable Hamilton system [2, 3] for the third-order spectral problem is rarely studied. In this paper, we consider the third-order eigenvalue problem

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_x = \begin{pmatrix} \frac{1}{2}i\xi & \mu_1 & \mu_2 \\ \mu_1^* & -\frac{1}{2}i\xi & 0 \\ \mu_2^* & 0 & -\frac{1}{2}i\xi \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \tag{1.1}$$

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and discuss the relation between the third-order spectral problem (1.1) and the following equation:

$$(1.2)\qquad \begin{cases} i\mu_{1t} = \mu_{1xx} - 2\mu_1\left(|\mu_1|^2 + |\mu_2|^2\right)\,, \\ i\mu_{2t} = \mu_{2xx} - 2\mu_2\left(|\mu_1|^2 + |\mu_2|^2\right)\,, \end{cases}$$

where $i = \sqrt{-1}$, $\mu_1 = \mu_1(x,t)$, $\mu_2 = \mu_2(x,t)$, and ξ is a complex eigenparameter of the eigenvalue problem (1.1). This system, often referred to in the literature as the coupled nonlinear Schrödinger equation, was used by Manakov for studying the propagation of the electric field in a waveguide [4]. Each equation governs the evolution of one of the components of the field transverse to the direction of propagation. Also it can be derived as a model for wave propagation under conditions similar to those where NLS applies and there are two wavetrains moving with nearly the same group velocity [5]. In recent years, this system was derived as a key model for light-wave propagation in optical fibers [6,7].

In the present paper, by using the adjoint representation of the eigenvalue problem (1.1), we obtain the evolution equation hierarchy and the Lax representation associated with eq. (1.2). On the real space R^{12}, a suitable symplectic structure, the Poisson bracket and Hamiltonian canonical equations are introduced, therefore, they are all written in the complex forms. By using the nonlinearization of the Lax pairs of the coupled nonlinear Schrödinger equation, a finite-dimensional completely integrable system of the complex form is given. Furthermore, the representation of the solution of the coupled nonlinear Schrödinger equation (1.2) is generated by using commutable flows of the finite-dimensional completely integrable systems.

2. – The complex representation of the symplectic structure and the Poisson bracket

We consider the symplectic structure for the fundamental coordinate function in the real space [8]

$$(2.1)\qquad \sum_{j=1}^{3}\sum_{k=1}^{2N} \mathrm{d}q_{jk}\wedge \mathrm{d}p_{jk} = \sum_{j=1}^{3} \mathrm{d}q_j \wedge \mathrm{d}p_j\,.$$

The Poisson bracket of the function H and F in the symplectic space $\left(R^{12N}, \sum_{j=1}^{3} \mathrm{d}q_j\wedge \mathrm{d}p_j\right)$ is defined as follows:

$$(2.2)\quad (H,F) = \sum_{j=1}^{3}\sum_{k=1}^{2N}\left(\frac{\partial H}{\partial q_{jk}}\frac{\partial F}{\partial p_{jk}} - \frac{\partial H}{\partial p_{jk}}\frac{\partial F}{\partial q_{jk}}\right) = \sum_{j=1}^{3}\left(\langle H_{q_j}, F_{p_j}\rangle - \langle H_{p_j}, F_{q_j}\rangle\right)$$

and the Hamilton canonical equations in the symplectic space are as follows:

$$(2.3)\qquad q_{j_t} = (q_j, H) = \frac{\partial H}{\partial p_j}\,,\qquad p_{j_t} = (p_j, H) = -\frac{\partial H}{\partial q_j}\,,\qquad j = 1,2,3\,.$$

Theorem 2.1

Set [9, 10]

$$\phi_{1k} = \frac{1}{\sqrt{2}}\left(-p_{1k} + iq_{3,N+k}\right), \ \phi_{2k} = -\frac{1}{\sqrt{2}}\left(p_{2k} + iq_{2,N+k}\right), \ \ \phi_{3k} = \frac{1}{\sqrt{2}}\left(-p_{3k} + iq_{1,N+k}\right),$$

$$\psi_{1k} = \frac{1}{\sqrt{2}}\left(q_{1k} - ip_{3,N+k}\right), \ \ \psi_{2k} = \frac{1}{\sqrt{2}}\left(q_{2k} + ip_{2,N+k}\right), \ \ \ \psi_{3k} = \frac{1}{\sqrt{2}}\left(q_{3k} - ip_{1,N+k}\right),$$

where $k = 1, 2, \cdots, N$.

Then the real representation for the symplectic stucture (2.1), the Poisson bracket (2.2) and the Hamilton canonical equation (2.3) is equivalent to the complex representation as follows:

$$\sum_{j=1}^{3}\sum_{k=1}^{2N} \mathrm{d}q_{jk} \wedge \mathrm{d}p_{jk} = \sum_{j=1}^{3}\sum_{k=1}^{N} \mathrm{d}\phi_{jk} \wedge \mathrm{d}\psi_{jk} + \sum_{j=1}^{3}\sum_{k=1}^{N} \mathrm{d}\phi^*_{jk} \wedge \mathrm{d}\psi^*_{jk}, \tag{2.4}$$

$$(H, F) = \sum_{j=1}^{3}\sum_{k=1}^{N}\left(\frac{\partial H}{\partial \phi_{jk}}\frac{\partial F}{\partial \psi_{jk}} - \frac{\partial H}{\partial \psi_{jk}}\frac{\partial F}{\partial \phi_{jk}} + \frac{\partial H}{\partial \phi^*_{jk}}\frac{\partial F}{\partial \psi^*_{jk}} - \frac{\partial H}{\partial \psi^*_{jk}}\frac{\partial F}{\partial \phi^*_{jk}}\right), \tag{2.5}$$

$$\phi_{jk_t} = (\phi_{jk}, H) = \frac{\partial H}{\partial \psi_{jk}}, \quad \psi_{jk_t} = (\psi_{jk}, H) = -\frac{\partial H}{\partial \phi_{jk}}, \tag{2.6}$$

$$\phi^*_{jk_t} = \left(\phi^*_{jk}, H\right) = \frac{\partial H}{\partial \psi^*_{jk}}, \quad \psi^*_{jk_t} = \left(\psi^*_{jk}, H\right) = -\frac{\partial H}{\partial \phi^*_{jk}},$$

$$j = 1, 2, 3, \qquad k = 1, 2, \cdots, N.$$

3. – The third-order spectral problem and the m–th-order evolution equation

We consider the spectral problem

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_x = \begin{pmatrix} \frac{1}{2}i\xi & \mu_1 & \mu_2 \\ \mu_1^* & -\frac{1}{2}i\xi & 0 \\ \mu_2^* & 0 & -\frac{1}{2}i\xi \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = M(\mu, \xi) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \tag{3.1}$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_{t_m} = \sum_{j=0}^{m} \begin{pmatrix} a_{11_j} & a_{12_j} & a_{13_j} \\ a_{21_j} & a_{22_j} & a_{23_j} \\ a_{31_j} & a_{32_j} & a_{33_j} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \xi^{m-j} = N_m(\mu, \xi) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}. \tag{3.2}$$

Set

$$V=\sum_{j=0}^{\infty}\begin{pmatrix} a_{11_j} & a_{12_j} & a_{13_j} \\ a_{21_j} & a_{22_j} & a_{23_j} \\ a_{31_j} & a_{32_j} & a_{33_j} \end{pmatrix}\xi^{-j}\,.$$

From the adjoint representation equation $V_x=[M,V]$, we have

$$\left\{\begin{array}{l} a_{11j_x}-\mu_1a_{21j}-\mu_2a_{31j}+\mu_1^*a_{12j}+\mu_2^*a_{13j}=0\,,\\ a_{22j_x}+\mu_1a_{21j}-\mu_1^*a_{12j}=0\,,\\ a_{33j_x}+\mu_2a_{31j}-\mu_2^*a_{13j}=0\,,\\ a_{23j_x}+\mu_2a_{21j}-\mu_1^*a_{13j}=0\,,\\ a_{32j_x}+\mu_1a_{31j}-\mu_2^*a_{12j}=0\,,\\ a_{12j_x}-ia_{12,j+1}-\mu_1a_{22j}-\mu_2a_{32j}+\mu_1a_{11j}=0\,,\\ a_{21j_x}+ia_{21,j+1}-\mu_1^*a_{11j}+\mu_1^*a_{22j}+\mu_2^*a_{23j}=0\,,\\ a_{13j_x}-ia_{13,j+1}-\mu_1a_{23j}-\mu_2a_{33j}+\mu_2a_{11j}=0\,,\\ a_{31j_x}+ia_{31,j+1}-\mu_2^*a_{11j}+\mu_1^*a_{32j}+\mu_2^*a_{33j}=0\,. \end{array}\right. \tag{3.3}$$

Set $\left(\partial\partial^{-1}=\partial^{-1}\partial=1\right)$

$$K=$$

$$\begin{pmatrix} \partial-2\mu_1\partial^{-1}\mu_1^*-\mu_2\partial^{-1}\mu_2^* & 2\mu_1\partial^{-1}\mu_1 & -\mu_1\partial^{-1}\mu_2^* & \mu_2\partial^{-1}\mu_1+\mu_1\partial^{-1}\mu_2 \\ 2\mu_1^*\partial^{-1}\mu_1^* & \partial-2\mu_1^*\partial^{-1}\mu_1-\mu_2^*\partial^{-1}\mu_2 & \mu_2^*\partial^{-1}\mu_1^*+\mu_1^*\partial^{-1}\mu_2^* & -\mu_1^*\partial^{-1}\mu_2 \\ -\mu_2\partial^{-1}\mu_1^* & \mu_2\partial^{-1}\mu_1+\mu_1\partial^{-1}\mu_2 & \partial-\mu_1\partial^{-1}\mu_1^*-2\mu_2\partial^{-1}\mu_2^* & 2\mu_2\partial^{-1}\mu_2 \\ \mu_2^*\partial^{-1}\mu_1^*+\mu_1^*\partial^{-1}\mu_2^* & -\mu_2^*\partial^{-1}\mu_1 & 2\mu_2^*\partial^{-1}\mu_2^* & \partial-2\mu_2^*\partial^{-1}\mu_2-\mu_1^*\partial^{-1}\mu_1 \end{pmatrix}$$

$$J=\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix},\qquad G_j=\begin{pmatrix} a_{12j} \\ a_{21j} \\ a_{13j} \\ a_{31j} \end{pmatrix},\qquad j=0,1,2\ldots,$$

then (3.3) becomes

$$KG_j=JG_{j+1},\qquad j=0,1,2\ldots. \tag{3.4}$$

We choose $a_{110}=i/2$, $a_{220}=a_{330}=-i/2$, $a_{120}=a_{210}=a_{130}=a_{310}=a_{230}=a_{320}=0$, and assume the constans of integration to be zero. In this way, the recursion relation (3.4) uniquely gives a series of polynomial functions with respect to $\mu,\mu_x,\cdots$. For example,

$$G_1=\begin{pmatrix} a_{121} \\ a_{211} \\ a_{131} \\ a_{311} \end{pmatrix}=\begin{pmatrix} \mu_1 \\ \mu_1^* \\ \mu_2 \\ \mu_2^* \end{pmatrix},\qquad G_2=\begin{pmatrix} a_{122} \\ a_{212} \\ a_{132} \\ a_{312} \end{pmatrix}=\begin{pmatrix} -i\mu_{1_x} \\ i\mu_{1_x}^* \\ -i\mu_{2_x} \\ i\mu_{2_x}^* \end{pmatrix}. \tag{3.5}$$

Theorem 3.1. The m-th–order evolution equation

$$\mu_{t_m}=\begin{pmatrix} \mu_1 \\ \mu_1^* \\ \mu_2 \\ \mu_2^* \end{pmatrix}=JG_{m+1}=KG_m,\qquad m=1,2\ldots \tag{3.6}$$

is equivalent to the Lax equation

$$M_{t_m} = (N_m)_x - [M, N_m] \,. \tag{3.7}$$

Proof. From the compatibility condition of (3.1) and (3.2), the theorem is proved. Example. From (3.5), we have the first-order evolution equation

$$\begin{pmatrix} \mu_1 \\ \mu_1^* \\ \mu_2 \\ \mu_2^* \end{pmatrix}_{t_1} = \begin{pmatrix} \mu_{1_x} \\ \mu_{1_x}^* \\ \mu_{2_x} \\ \mu_{2_x}^* \end{pmatrix}$$

and the second-order evolution equation

$$\begin{pmatrix} \mu_1 \\ \mu_1^* \\ \mu_2 \\ \mu_2^* \end{pmatrix}_{t_2} = \begin{pmatrix} -\mu_{1_{xx}} + 2i\mu_1 \left(|\mu_1|^2 + |\mu_2|^2\right) \\ i\mu_{1_{xx}}^* - 2i\mu_1^* \left(|\mu_1|^2 + |\mu_2|^2\right) \\ -\mu_{2_{xx}} + 2i\mu_2 \left(|\mu_1|^2 + |\mu_2|^2\right) \\ i\mu_{2_{xx}}^* - 2i\mu_2^* \left(|\mu_1|^2 + |\mu_2|^2\right) \end{pmatrix} . \tag{3.8}$$

Obviously, (3.8) is equivalent to (1.2).

4. – The functional gradient of the eigenvalue

Consider the third-order eigenvalue problem

$$\begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}_x = \begin{pmatrix} \frac{1}{2}i\xi_j & \mu_1 & \mu_2 \\ \mu_1^* & -\frac{1}{2}i\xi_j & 0 \\ \mu_2^* & 0 & -\frac{1}{2}i\xi_j \end{pmatrix} \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix} = M\left(\xi_j, \mu\right) \begin{pmatrix} \phi_{1j} \\ \phi_{2j} \\ \phi_{3j} \end{pmatrix}, \tag{4.1}$$

$j = 1, 2 \cdots N$

and the adjoint eigenvalue problem

$$\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}_x = -M^T\left(\xi_j, \mu\right) \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \\ \psi_{3j} \end{pmatrix}, \quad j = 1, 2 \cdots N \,, \tag{4.2}$$

where ξ_j is an eigenvalue of (4.1) and (4.2), $\left(\overline{\Phi}_j, \overline{\Psi}_j\right)^T = \left(\left(\phi_{1j}, \phi_{2j}, \phi_{3j}\right)^T, \left(\psi_{1j}, \psi_{2j}, \psi_{3j}\right)^T\right)$ is the associated eigenfunction.

Then the conjugate representations of (4.1) and (4.2) are

$$\begin{pmatrix} \phi_{1j}^* \\ \phi_{2j}^* \\ \phi_{3j}^* \end{pmatrix}_x = \begin{pmatrix} -\frac{1}{2}i\xi_j^* & \mu_1^* & \mu_2^* \\ \mu_1 & \frac{1}{2}i\xi_j^* & 0 \\ \mu_2 & 0 & \frac{1}{2}i\xi_j^* \end{pmatrix} \begin{pmatrix} \phi_{1j}^* \\ \phi_{2j}^* \\ \phi_{3j}^* \end{pmatrix} = M^*\left(\xi_j, \mu\right) \begin{pmatrix} \phi_{1j}^* \\ \phi_{2j}^* \\ \phi_{3j}^* \end{pmatrix}, \tag{4.3}$$

$j = 1, 2 \cdots N,$

$$(4.4)\quad \begin{pmatrix} \psi_{1j}^* \\ \psi_{2j}^* \\ \psi_{3j}^* \end{pmatrix}_x = \begin{pmatrix} \frac{1}{2}i\xi_j^* & -\mu_1 & -\mu_2 \\ -\mu_1^* & -\frac{1}{2}i\xi_j^* & 0 \\ -\mu_2^* & 0 & -\frac{1}{2}i\xi_j^* \end{pmatrix} \begin{pmatrix} \phi_{1j}^* \\ \phi_{2j}^* \\ \phi_{3j}^* \end{pmatrix} = -M^{T^*}(\xi_j,\mu) \begin{pmatrix} \psi_{1j}^* \\ \psi_{2j}^* \\ \psi_{3j}^* \end{pmatrix},$$

$j = 1, 2 \cdots N$.

Theorem 4.1. Let ξ_j be an eigenvalue of (4.1) and (4.2), then

$$(4.5)\quad \nabla\xi_j = \begin{pmatrix} \frac{\delta\xi_j}{\delta\mu_1^*} \\ \frac{\delta\xi_j}{\delta\mu_1} \\ \frac{\delta\xi_j}{\delta\mu_2^*} \\ \frac{\delta\xi_j}{\delta\mu_2} \end{pmatrix} = \left(\int_\Omega \left(-\frac{1}{2}\phi_{1j}\psi_{1j} + \frac{1}{2}\phi_{2j}\psi_{2j} + \frac{1}{2}\phi_{3j}\psi_{3j}\right) \mathrm{d}x\right)^{-1} \begin{pmatrix} \phi_{1j}\psi_{2j} \\ \phi_{2j}\psi_{1j} \\ \phi_{1j}\psi_{3j} \\ \phi_{3j}\psi_{1j} \end{pmatrix}$$

and

$$(4.6)\quad \nabla\xi_j^* = \begin{pmatrix} \frac{\delta\xi_j^*}{\delta\mu_1} \\ \frac{\delta\xi_j^*}{\delta\mu_1^*} \\ \frac{\delta\xi_j^*}{\delta\mu_2} \\ \frac{\delta\xi_j^*}{\delta\mu_2^*} \end{pmatrix} = \left(\int_\Omega \left(\frac{1}{2}\phi_{1j}^*\psi_{1j}^* - \frac{1}{2}\phi_{2j}^*\psi_{2j}^* - \frac{1}{2}\phi_{3j}^*\psi_{3j}^*\right) \mathrm{d}x\right)^{-1} \begin{pmatrix} \phi_{1j}^*\psi_{2j}^* \\ \phi_{2j}^*\psi_{1j}^* \\ \phi_{1j}^*\psi_{3j}^* \\ \phi_{3j}^*\psi_{1j}^* \end{pmatrix},$$

$j = 1, 2 \cdots N$.

Proof. Direct compute from (4.1) to (4.4).

Then from (4.5) and (4.6), through calculation, we obtain the conclusions as follows: Corollary 4.1

1) $K\nabla\xi_j = \xi_j J\nabla\xi_j, \quad j = 1, 2 \cdots N;$
2) $K\nabla\xi_j^* = \xi_j^* J\nabla\xi_j^*, \quad j = 1, 2 \cdots N.$

Corollary 4.2

$$(4.7)\quad K \begin{pmatrix} \phi_{1j}\psi_{2j} + \phi_{1j}^*\psi_{2j}^* \\ \phi_{2j}\psi_{1j} + \phi_{2j}^*\psi_{1j}^* \\ \phi_{1j}\psi_{3j} + \phi_{1j}^*\psi_{3j}^* \\ \phi_{3j}\psi_{1j} + \phi_{3j}^*\psi_{1j}^* \end{pmatrix} = J \begin{pmatrix} \xi_j\phi_{1j}\psi_{2j} + \xi_j^*\phi_{1j}^*\psi_{2j}^* \\ \xi_j\phi_{2j}\psi_{1j} + \xi_j^*\phi_{2j}^*\psi_{1j}^* \\ \xi_j\phi_{1j}\psi_{3j} + \xi_j^*\phi_{1j}^*\psi_{3j}^* \\ \xi_j\phi_{3j}\psi_{1j} + \xi_j^*\phi_{3j}^*\psi_{1j}^* \end{pmatrix}.$$

5. – A complex complete integrable system in the Liouville sense

Set $\Phi_j = \left(\phi_{j1}, \cdots, \phi_{jN}, \phi_{j1}^*, \cdots, \phi_{jN}^*\right)$, $\Psi_j = \left(\psi_{j1}, \cdots, \psi_{jN}, \psi_{j1}^*, \cdots, \psi_{jN}^*\right)$, $\Lambda = \mathrm{diag}\left(\xi_1, \cdots, \xi_N, \xi_1^*, \cdots, \xi_N^*\right)$, $j = 1, 2, 3$.

We consider the constraint relations between the potential functions and the eigenfunctions

$$(5.1)\quad \mu = \begin{pmatrix} \mu_1 \\ \mu_1^* \\ \mu_2 \\ \mu_2^* \end{pmatrix} = \begin{pmatrix} i\langle\Phi_1, \Psi_2\rangle \\ i\langle\Phi_2, \Psi_1\rangle \\ i\langle\Phi_1, \Psi_3\rangle \\ i\langle\Phi_3, \Psi_1\rangle \end{pmatrix} = G_1 .$$

From (3.4) and (5.1), we have

$$G_j = \begin{pmatrix} i\langle \Lambda^{j-1}\Phi_1, \Psi_2\rangle \\ i\langle \Lambda^{j-1}\Phi_2, \Psi_1\rangle \\ i\langle \Lambda^{j-1}\Phi_1, \Psi_3\rangle \\ i\langle \Lambda^{j-1}\Phi_3, \Psi_1\rangle \end{pmatrix}, \quad j = 1, 2\ldots \tag{5.2}$$

and so we derive

$$\begin{cases} a_{11j} = i\langle \Lambda^{j-1}\Phi_1, \Psi_1\rangle \\ a_{22j} = i\langle \Lambda^{j-1}\Phi_2, \Psi_2\rangle \\ a_{33j} = i\langle \Lambda^{j-1}\Phi_3, \Psi_3\rangle \\ a_{23j} = i\langle \Lambda^{j-1}\Phi_2, \Psi_3\rangle \\ a_{32j} = i\langle \Lambda^{j-1}\Phi_3, \Psi_2\rangle \end{cases}, \quad j = 1, 2\ldots \tag{5.3}$$

Theorem 5.1. (4.1), (4.2), (4.3) and (4.4) can be written as follows:

$$\begin{cases} \Phi_{1x} = \dfrac{\partial H_0}{\partial \Psi_1}, \quad \Phi_{2x} = \dfrac{\partial H_0}{\partial \Psi_2}, \quad \Phi_{3x} = \dfrac{\partial H_0}{\partial \Psi_3}, \\ \Psi_{1x} = -\dfrac{\partial H_0}{\partial \Phi_1}, \quad \Psi_{2x} = -\dfrac{\partial H_0}{\partial \Phi_2}, \quad \Psi_{3x} = -\dfrac{\partial H_0}{\partial \Phi_3}, \end{cases} \tag{5.4}$$

where

$$H_0 = \frac{i}{2}\langle \Lambda\Phi_1, \Psi_1\rangle - \frac{i}{2}\langle \Lambda\Phi_2, \Psi_2\rangle - \frac{i}{2}\langle \Lambda\Phi_3, \Psi_3\rangle + i\langle \Lambda\Phi_1, \Psi_2\rangle\langle \Lambda\Phi_2, \Psi_1\rangle + \\ + i\langle \Lambda\Phi_1, \Psi_3\rangle\langle \Lambda\Phi_3, \Psi_1\rangle.$$

Proof. From (5.1), we obtain (5.4).
Set

$$\overline{V} = \begin{pmatrix} V & 0 \\ 0 & V^* \end{pmatrix} \qquad \overline{M} = \begin{pmatrix} M & 0 \\ 0 & M^* \end{pmatrix}.$$

By using $V_x = [M, V]$ and $V_x^* = [M^*, V^*]$, we have $\overline{V_x} = [\overline{M}, \overline{V}]$. Therefore we derive $\left(\overline{V}^n\right)_x = [\overline{M}, \overline{V}^n]$ $(n \geq 1)$. Then tr $\left(\overline{V}^2\right)_x = \operatorname{tr}[\overline{M}, \overline{V}^2] = 0$.

Set $F = (1/2)\operatorname{tr}\overline{V^2} = \sum_{m=0}^{\infty} F_m \xi^{-m}$, we obtain the following expressions:

$$F_m = \\ \sum_{j=0}^m \left(\tfrac{1}{2}a_{11j}a_{11,m-j} + \tfrac{1}{2}a_{22j}a_{22,m-j} + \tfrac{1}{2}a_{33j}a_{33,m-j} + a_{12j}a_{21,m-j} + a_{13j}a_{31,m-j} + a_{23j}a_{32,m-j}\right) + \\ + \sum_{j=0}^m \left(\tfrac{1}{2}a^*_{11j}a^*_{11,m-j} + \tfrac{1}{2}a^*_{22j}a^*_{22,m-j} + \tfrac{1}{2}a^*_{33j}a^*_{33,m-j} + a^*_{12j}a^*_{21,m-j} + a^*_{13j}a^*_{31,m-j} + a^*_{23j}a^*_{32,m-j}\right),$$

$$F_0 = -\frac{3}{8}, \quad F_1 = 0, \quad F_2 = iH_0,$$

$$F_{m+1} = -\frac{1}{2}\langle \Lambda^m\Phi_1, \Psi_1\rangle + \frac{1}{2}\langle \Lambda^m\Phi_2, \Psi_2\rangle + \frac{1}{2}\langle \Lambda^m\Phi_3, \Psi_3\rangle - \\ -\frac{1}{2}\sum_{j=1}^m \left(\langle \Lambda^{j-1}\Phi_1, \Psi_1\rangle\langle \Lambda^{m-j}\Phi_1, \Psi_1\rangle + \langle \Lambda^{j-1}\Phi_2, \Psi_2\rangle\langle \Lambda^{m-j}\Phi_2, \Psi_2\rangle + \langle \Lambda^{j-1}\Phi_3, \Psi_3\rangle\langle \Lambda^{m-j}\Phi_3, \Psi_3\rangle\right) - \\ - \sum_{j=1}^m \left(\langle \Lambda^{j-1}\Phi_1, \Psi_2\rangle\langle \Lambda^{m-j}\Phi_2, \Psi_1\rangle + \langle \Lambda^{j-1}\Phi_1, \Psi_3\rangle\langle \Lambda^{m-j}\Phi_3, \Psi_1\rangle + \langle \Lambda^{j-1}\Phi_2, \Psi_3\rangle\langle \Lambda^{m-j}\Phi_3, \Psi_2\rangle\right).$$

The time evolution equation of the eigenfunction

$$\left(\begin{array}{c}\Phi_{1j}\\ \Phi_{2j}\\ \Phi_{3j}\end{array}\right)_{t_m}=N_m\left(\mu,\xi\right)\left(\begin{array}{c}\Phi_{1j}\\ \Phi_{2j}\\ \Phi_{3j}\end{array}\right),\quad j=1,2\cdots N \tag{5.5}$$

and the time evolution equation of the adjoint eigenfunction

$$\left(\begin{array}{c}\Psi_{1j}\\ \Psi_{2j}\\ \Psi_{3j}\end{array}\right)_{t_m}=-N_m^T\left(\mu,\xi\right)\left(\begin{array}{c}\Psi_{1j}\\ \Psi_{2j}\\ \Psi_{3j}\end{array}\right),\quad j=1,2\ldots N, \tag{5.6}$$

then the conjugate representations of (5.5) and (5.6) are

$$\left(\begin{array}{c}\Phi_{1j}^*\\ \Phi_{2j}^*\\ \Phi_{3j}^*\end{array}\right)_{t_m}=N_m^*\left(\mu,\xi\right)\left(\begin{array}{c}\Phi_{1j}^*\\ \Phi_{2j}^*\\ \Phi_{3j}^*\end{array}\right),\quad j=1,2,\ldots N, \tag{5.7}$$

$$\left(\begin{array}{c}\Psi_{1j}^*\\ \Psi_{2j}^*\\ \Psi_{3j}^*\end{array}\right)_{t_m}=-N_m^{T^*}\left(\mu,\xi\right)\left(\begin{array}{c}\Psi_{1j}^*\\ \Psi_{2j}^*\\ \Psi_{3j}^*\end{array}\right),\quad j=1,2\ldots N. \tag{5.8}$$

Let

$$H_m=F_{m+1},\text{ then } H_m^*=H_m,\quad m=1,2,\ldots$$

Theorem 5.2. (5.5), (5.6), (5.7) and (5.8) can be written as follows:

$$\begin{cases}\Phi_{1_{t_m}}=\dfrac{\partial H_m}{\partial\Psi_1}, & \Phi_{2_{t_m}}=\dfrac{\partial H_m}{\partial\Psi_2}, & \Phi_{3_{t_m}}=\dfrac{\partial H_m}{\partial\Psi_3},\\ \Psi_{1_{t_m}}=-\dfrac{\partial H_m}{\partial\Phi_1}, & \Psi_{2_{t_m}}=-\dfrac{\partial H_m}{\partial\Phi_2}, & \Psi_{3_{t_m}}=-\dfrac{\partial H_m}{\partial\Phi_3}.\end{cases} \tag{5.9}$$

Proof. From (5.2), (5.3) and Theorem 2.1, we obtain (5.9).

We know that $F=(1/2)\,\mathrm{tr}\overline{V}^{\,2}$ is also a generating function of integrals of motion for (5.9). Therefore [11,12] $\{H_m,H_n\}=\{F_{m+1},F_{n+1}\}=\frac{\partial F_{n+1}}{\partial t_{m+1}}\equiv 0,\ m,n\geq 0.$

Corollary 5.1

The Hamiltonian systems (5.4) and (5.9) are complete integrable systems in the Liouville sense.

Theorem 5.3. Let

$$\Phi_j=\left(\phi_{j1}\left(x,t_m\right),\cdots,\phi_{jN}\left(x,t_m\right),\phi_{j1}^*\left(x,t_m\right),\cdots,\phi_{jN}^*\left(x,t_m\right)\right)^T,$$

$$\Psi_j=\left(\psi_{j1}\left(x,t_m\right),\cdots,\psi_{jN}\left(x,t_m\right),\psi_{j1}^*\left(x,t_m\right),\cdots,\psi_{jN}^*\left(x,t_m\right)\right)^T$$

be an involutive solution of the systems (5.4) and (5.9), $\mu=(\mu_1\ \ \mu_1^*\ \ \mu_2\ \ \mu_2^*)^T$ and (Φ_j,Ψ_j) satisfy (5.1), then

1) The Hamiltonian equations (5.4) and (5.9) reduce the spatial part and time part of the Lax pair for the m-th–order equation (3.6).

2) $\mu = (\mu_1 \ \ \mu_1^* \ \ \mu_2 \ \ \mu_2^*)^T$ satisfies the m-th–order equation (3.6).

6. – Conclusions

In this paper, the Lax representation of the coupled nonlinear Schrödinger equation hierarchy is discussed. Based on the constrained relations between the potential and eigenfunctions of the spectral problem, we get a completely integrable system of complex form in the Liouville sense. Moreover, the solutions of the equation are derived by using the involutive solution of the completely integrable Hamiltonian system.

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