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Proper curvature collineations in Bianchi type-II space-times(*)

G. Shabbir(**)

Faculty of Engineering Sciences, GIK Institute of Engineering Sciences and Technology Topi Swabi, NWFP, Pakistan

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Summary. — A study of Bianchi type-II space-times according to their proper curvature collineations is given by using the rank of the 6×6 Riemann matrix and direct integration techniques. It is shown that there exist only two cases when the above space-time admits proper curvature collineations and they form an infinite-dimensional vector space.

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1. – Introduction

Over the past few years there has been much interest in the classification of the space-times in terms of curvature collineations (CCs). These curvature collineations are vector fields, along which the Lie derivative of the Riemann tensor is zero. Different approaches [1-13] were adopted to study CCs. The aim of this paper is to find the existence of proper curvature collineations in Bianchi type-II space-times. In this paper an aproach, which is given in [4], is adopted to study proper curvature collineations in Bianchi type-II space-times by using the rank of the 6×6 Riemann matrix and direct integration techinques. Throughout M is representing the four-dimensional, connected, Hausdorff space-time manifold with the Lorentz metric g of signature $(-,+,+,+)$. The curvature tensor associated with g, through Levi-Civita connection, is denoted in component form by $R^a{}_{bcd}$. The usual covariant, partial and Lie derivatives are denoted by a semicolon, a comma and the symbol L, respectively. Round and square brackets denote the usual symmetrization and skew-symmetrization, respectively.

Since the curvature tensor is an important object in differential geometry and, in particular, in general relativity, thus the study of its symmetries, *i.e.* curvature collineations is important.

(*) The author of this paper has agreed to not receive the proofs for correction.
(**) E-mail: `shabbir@giki.edu.pk`

Any vector field Z on M can be decomposed as

$$Z_{a;b} = \frac{1}{2}h_{ab} + F_{ab}\,, \tag{1}$$

where $h_{ab}(= h_{ba}) = L_Z g_{ab}$ is a symmetric and $F_{ab}(= -F_{ba})$ is a skew-symmetric tensor on M. If $h_{ab;c} = 0$, Z is said to be *affine* and further satisfies $h_{ab} = 2cg_{ab}$, $c \in R$, then X is said to be *homothetic* (and *Killing* if $c = 0$). The vector field Z is said to be proper affine if it is not a homothetic vector field and also Z is said to be a proper homothetic vector field if it is not a Killing vector field.

A vector field Z on M is called a curvature collineation (CC) if it satisfies [14]

$$L_Z R^a{}_{bcd} = 0\,, \tag{2}$$

or, equivalently,

$$R^a{}_{bcd;e}Z^e + R^a{}_{ecd}Z^e{}_{;b} + R^a{}_{bed}Z^e{}_{;c} + R^a{}_{bce}Z^e{}_{;d} - R^e{}_{bcd}Z^a{}_{;e} = 0.$$

The vector field Z is said to be proper CC if it is not affine [4] on M.

2. – Classification of the Riemann tensor

In this section we will classify the Riemann tensor in terms of its rank and bivector decomposition.

The rank of the Riemann tensor is the rank of the 6×6 symmetric matrix derived in a well-known way [4]. The rank of the Riemann tensor at p is the rank of the linear map f which maps the vector space of all bivectors F at p to itself and is defined by $f : F^{ab} \rightarrow R^{ab}{}_{cd}F^{cd}$. And also define the subspace N_p of the tangent space T_pM consisting of those members k of T_pM which satisfy the relation

$$R_{abcd}k^d = 0. \tag{3}$$

Then the Riemann tensor at p satisfies exactly one of the following algebraic conditions [4].

Class B

The rank is 2 and the range of f is spanned by the dual pair of non-null simple bivectors and $\dim N_p = 0$. The Riemann tensor at p takes the form

$$R_{abcd} = \alpha F_{ab}F_{cd} + \beta \overset{*}{F}_{ab}\overset{*}{F}_{cd}\,, \tag{4}$$

where F and its dual $\overset{*}{F}$ are the (unique up to scaling) simple non-null spacelike and timelike bivectors in the range of f, respectively and $\alpha, \beta \in R$.

Class C

The rank is 2 or 3 and there exists a unique (up to scaling) solution say, k of (3) (and so $\dim N_p = 1$). The Riemann tensor at p takes the form

$$R_{abcd} = \sum_{i,j=1}^{3} \alpha_{ij} F^i{}_{ab}F^j{}_{cd}\,, \tag{5}$$

where $\alpha_{ij} \in R$ for all i, j and $F^i{}_{ab}k^b = 0$ for each of the bivectors F^i which span the range of f.

Class D

Here the rank of the curvature matrix is 1. The range of the map f is spanned by a single bivector F, say, which has to be simple because the symmetry of the Riemann tensor $R_{a[bcd]} = 0$ means $F_{a[b}F_{cd]} = 0$. Then it follows from a standard result that F is simple. The curvature tensor admits exactly two independent solutions k, u of eq. (3) so that $\dim N_p = 2$. The Riemann tensor at p takes the form

$$R_{abcd} = \alpha F_{ab} F_{cd}\,, \tag{6}$$

where $\alpha \in R$ and F is a simple bivector with the blade orthogonal to k and u.

Class O

The rank of the curvature matrix is 0 (so that $R_{abcd} = 0$) and $\dim N_p = 4$.

Class A

The Riemann tensor is said to be of class A at p if it is not of class B, C, D or O. Here always $\dim N_p = 0$.

3. – Curvature collineations

A vector field Z is said to CC if it satisfies eq. (2). The algebraic classification of the Riemann tensor at p is given in sect. **2** and at p it satisfies exactly one of the algebraic conditions, *i.e.* A, B, C, D or O. Curvature collineations for the class C will be discussed in the following. A study of the CCs for the classes A, B, D and O can be found in [4,13].

Class C

Here we will focus on a space-time M, which is everywhere of class C. If M is of class C, then there exists a unique (up to a multiple) solution of (3), say k at each $p \in M$. If the vector field k is timelike or spacelike, then we say M is of class C_T or C_S, respectively. Here, we consider the special case [9] when k is covariantly constant (the general solution of (3) is considered in [9]).

First suppose that M is everywhere of class C_T. The rank of the 6×6 Riemann matrix is two or three and also there exists a unique up to a sign nowhere zero timelike vector field $t_a = t_{,a}$ satisfying $t_{a;b} = 0$ and also $t_a t^a = -1$. From the Ricci identity, $R^a{}_{bcd}t_a = 0$. It follows from [4] that M is locally metric decomposable and, at any $p \in M$, thus one can choose a coordinate neighborhood U_p in which the line element is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \quad (\alpha, \beta = 1, 2, 3), \tag{7}$$

where $g_{\alpha\beta}$ depends only on x^γ ($\gamma = 1, 2, 3$). The above space-time is clearly $1 + 3$ decomposable. It follows from [4] that the CCs in this case are

$$Z = f(t)\frac{\partial}{\partial t} + Z', \tag{8}$$

where $f(t)$ is an arbitrary function of t and Z' is a homothetic vector field in the induced geometry on each of the three-dimensional submanifolds of constant t. Clearly CCs in (10) form an infinite-dimensional vector space.

The case when M is everywhere of class C_S is exactly the same. If M is C_S, there exists a spacelike vector field which is a solution of (3) at each $p \in M$. We are also assuming that the spacelike vector field is covariantly constant. In this case the induced metric in the hypersurfaces is Lorentzian. The CCs in this case are

$$Z = f(x)\frac{\partial}{\partial x} + Z', \tag{9}$$

where $f(x)$ is an arbitrary function of x and Z' is a homothetic vector field in the induced geometry on each of the three-dimensional submanifolds of constant x.

4. – Main results

Consider Bianchi type-II space-time in the usual coordinate system (t, x, y, z) (labeled by x^0, x^1, x^2, x^3, respectively) with the line element [15]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + A^2(t)\mathrm{d}x^2 + e^{2qx}\big(X^2(t)\mathrm{d}y^2 + Y^2(t)\mathrm{d}z^2\big), \tag{10}$$

where $A(t)$, $X(T)$ and $Y(t)$ are nowhere zero functions of t only and q is a non-zero constant on M (if $q = 0$, then the above space-time becomes Bianchi type-I and their CCs are given in [1]). The above space-time admits three linearly independent Killing vector fields which are $\frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial x} - qy\frac{\partial}{\partial y} - qz\frac{\partial}{\partial z}$. The non-zero independent components of the Riemann tensor are

$$\begin{aligned}
&R_{0101} = -A\ddot{A} \equiv \alpha_1, \qquad R_{0202} = -e^{2qx}X\ddot{X} \equiv \alpha_2,\\
&R_{0303} = -e^{2qx}Y\ddot{Y} \equiv \alpha_3, \qquad R_{1212} = e^{2qx}A^2X^2\left(\frac{\dot{A}\dot{X}}{AX} - \frac{q^2}{A^2}\right) \equiv \alpha_4,\\
&R_{1313} = e^{2qx}A^2Y^2\left(\frac{\dot{A}\dot{Y}}{AY} - \frac{q^2}{A^2}\right) \equiv \alpha_5, \qquad R_{2323} = e^{4qx}X^2Y^2\left(\frac{\dot{X}\dot{Y}}{XY} - \frac{q^2}{A^2}\right) \equiv \alpha_6,\\
&R_{0212} = qe^{2qx}X^2\left(\frac{\dot{A}}{A} - \frac{\dot{X}}{X}\right) \equiv \alpha_7, \qquad R_{0313} = qe^{2qx}Y^2\left(\frac{\dot{A}}{A} - \frac{\dot{Y}}{Y}\right) \equiv \alpha_8.
\end{aligned}$$

Let us write the curvature tensor with components R_{abcd} at p as a 6×6 symmetric matrix in the well-known way [16]

$$R_{abcd} = \begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & 0 & 0\\
0 & \alpha_2 & 0 & \alpha_7 & 0 & 0\\
0 & 0 & \alpha_3 & 0 & \alpha_8 & 0\\
0 & \alpha_7 & 0 & \alpha_4 & 0 & 0\\
0 & 0 & \alpha_8 & 0 & \alpha_5 & 0\\
0 & 0 & 0 & 0 & 0 & \alpha_6
\end{pmatrix}. \tag{11}$$

The possible rank of the 6×6 Riemann matrix is six, five, four or three. Rank two and one is not possible for the following reason: suppose the rank of the 6×6 Riemann matrix is two. Then there is only two non-zero rows or columns in (11). If we set any four rows or columns identically zero in (11) with judicious choice of A, X and Y, this forces the rank of the 6×6 Riemann matrix to be zero thus giving a contradiction. A similar argument will also apply to the rank one. We are only interested in those cases when the rank of the 6×6 Riemann matrix is less than or equal to three. Since we know from theorem [5] that when the rank of the 6×6 Riemann matrix is greater than three there exist no proper CCs. Thus there exist only two possibilities:

(A1) Rank $= 3$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_7 = \alpha_8 = 0$, $\alpha_4 \neq 0$, $\alpha_5 \neq 0$ and $\alpha_6 \neq 0$.

(A2) Rank $= 3$, $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_7 = 0$, $\alpha_3 \neq 0$, $\alpha_5 \neq 0$, $\alpha_6 \neq 0$ and $\alpha_8 \neq 0$.

We will consider each case in turn.

Case A1

In this case $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_7 = \alpha_8 = 0$, $\alpha_4 \neq 0$, $\alpha_5 \neq 0$, $\alpha_6 \neq 0$, the rank of the 6×6 Riemann matrix is 3 and there exists a unique (up to a multiple) nowhere zero timelike vector field $t_a = t_{,a}$ solution of eq. (3) and $t_{a;b} \neq 0$. From the above constraints we have $A = a_1 t + a_2$, $X = b_1 t + b_2$, $Y = c_1 t + c_2$, $A = k_1 X$, $Y = k_2 A$ and $\dfrac{a_2 b_1}{b_2} \neq \pm q$, where $a_1, a_2, b_1, b_2, c_1, c_2, k_1, k_2 \in R \backslash \{0\}$. The line element can, after a recaling of x, y and z, be written in the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (b_1 t + b_2)^2 \big(\mathrm{d}x^2 + e^{2qx} \big(\mathrm{d}y^2 + \mathrm{d}z^2\big)\big). \tag{12}$$

Substituting the above information into the CC equations [17] and after some calculation one finds CCs in this case are

$$\begin{aligned} Z^0 = N(t), \qquad Z^1 = c_1, \qquad & Z^2 = -qyc_1 - zc_3 + c_2, \\ & Z^3 = -qzc_1 + yc_3 + c_4, \end{aligned} \tag{13}$$

where $c_1, c_2, c_3, c_4 \in R$ and $N(t)$ is an arbitrary function of t only. One can write the above equation (13), after subtracting Killing vector fields as

$$Z = (N(t), 0, 0, 0). \tag{14}$$

Clearly, in this case proper CCs form an infinite-dimensional vector space.

Now consider the subcase when $b_1 = 0$ in (12) the line element can, after a recaling of x, y and z, be written in the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \big(\mathrm{d}x^2 + e^{2qx} \big(\mathrm{d}y^2 + \mathrm{d}z^2\big)\big), \tag{15}$$

where $q \neq 0$ or 1. The space-time is clearly $1 + 3$ decomposable. In this case the rank of the 6×6 Riemann matrix is 3 and there exists a unique (up to a multiple) nowhere zero timelike vector field $t_a = t_{,a}$ such that $t_{a;b} = 0$ (and from the Ricci identity $R^a{}_{bcd} t_a = 0$). The CCs in this case [4] are

$$Z = N(t) \frac{\partial}{\partial t} + Z', \tag{16}$$

where $N(t)$ is an arbitrary function of t and Z' is a homothetic vector field in the induced geometry on each of the three-dimensional submanifolds of constant t. The completion of case A1 necessitates finding a homothetic vector field in the induced geometry of the submanifolds of constant t. The induced metric $g_{\alpha\beta}$ (where $\alpha, \beta = 1, 2, 3$) with non-zero components is given by

$$g_{11} = 1,\ g_{22} = e^{2qx},\ g_{33} = e^{2qx}. \tag{17}$$

A vector field Z' is called homothetic vector field if it satisfies $L_{Z'} g_{\alpha\beta} = 2c g_{\alpha\beta}$, where $c \in R$. One can expand by using (14) to get

$$Z^1{}_{,1} = c, \tag{18}$$
$$Z^1{}_{,2} + e^{2qx} Z^2{}_{,1} = 0, \tag{19}$$
$$Z^1{}_{,3} + e^{2qx} Z^3{}_{,1} = 0, \tag{20}$$
$$qZ^1 + Z^2{}_{,2} = c, \tag{21}$$
$$Z^2{}_{,3} + Z^3{}_{,2} = 0, \tag{22}$$
$$qZ^1 + Z^3{}_{,3} = c. \tag{23}$$

Equations (18), (19) and (20) give

$$\begin{aligned} &Z^1 = cx + A^1(y,z), \qquad Z^2 = \frac{1}{2q} e^{-2qx} A^1{}_y(y,z) + A^2(y,z), \\ &Z^3 = \frac{1}{2q} e^{-2qx} A^1{}_z(y,z) + A^3(y,z), \end{aligned}$$

where $A^1(y,z)$, $A^2(y,z)$ and $A^3(y,z)$ are functions of integration. If one proceeds further after a straightforward calculation, one can find that proper homothetic vector fields only exist if $q = 0$, which is not possible. So homothetic vector fields in the induced gemetry are the Killing vector fields which are

$$\begin{aligned} Z^1 &= yc_1 + zc_3 + c_5, \\ Z^2 &= \frac{1}{2q} e^{-2qx} c_1 + \frac{q}{2}(z^2 - y^2)c_1 - qyzc_3 - qyc_5 - zc_6 + c_2, \\ Z^3 &= \frac{1}{2q} e^{-2qx} c_3 + \frac{q}{2}(y^2 - z^2)c_3 - qyzc_1 - qzc_5 + yc_6 + c_4, \end{aligned} \tag{24}$$

where $c_1, c_2, c_3, c_4, c_5, c_6 \in R$. CCs in this case are (from (16) and (24))

$$\begin{aligned} &Z^0 = N(t), \qquad Z^1 = yc_1 + zc_3 + c_5, \\ &Z^2 = \frac{1}{2q} e^{-2qx} c_1 + \frac{q}{2}(z^2 - y^2)c_1 - qyzc_3 - qyc_5 - zc_6 + c_2, \\ &Z^3 = \frac{1}{2q} e^{-2qx} c_3 + \frac{q}{2}(y^2 - z^2)c_3 - qyzc_1 - qzc_5 + yc_6 + c_4, \end{aligned} \tag{25}$$

where $N(t)$ is an arbitrary function of t. The above equation (25) can be written as eq. (14) after subtracting Killing vector fields.

Case A2

In this case $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_7 = 0$, $\alpha_3 \neq 0$, $\alpha_5 \neq 0$, $\alpha_6 \neq 0$, $\alpha_8 \neq 0$, the rank of the 6×6 Riemann matrix is 3 and there exist no non-trival solutions of eq. (3). Equations

$\alpha_1 = \alpha_2 = \alpha_4 = \alpha_7 = 0 \Rightarrow A = a_1 t + a_2$, $B = b_1 t + b_2$, $q = \pm a_1$ and $X = kA$, respectively, where $a_1, a_2, b_1, b_2, k \in R(a_1 \neq 0, b_1 \neq 0, k \neq 0)$. The line element can, after a rescaling of y, be written in the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (a_1 t + a_2)^2 \mathrm{d}x^2 + e^{2qx}\big((a_1 t + a_2)^2 \mathrm{d}y^2 + Y^2(t)\mathrm{d}z^2\big). \tag{26}$$

This case belongs to the general class and so no proper CCs exist [4, 13].

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