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Symmetry reductions for 2-dimensional non-linear wave equation

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Summary. — Lie symmetry method is used to find symmetry reductions for the non-linear wave equation $u_{tt} = u^n(u_{xx} + u_{yy})$. A set of symmetries and Lie algebra are found and reduction under each 2-dimensional sub-algebra is presented.

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1. – Introduction

Finding exact analytical solutions of nonlinear partial differential equations (PDEs) is one of the challenging problems in applied mathematics. Whilst there exist methods in the literature which can be used to tackle such problems, a large number of problems are wide open for an analytical study using the standard or conventional methods. In particular, the case of nonlinear PDEs poses the most difficult challenge. In such cases, approaches involving transformation of the given differential equation to a simpler equation while preserving the invariance of the original equation is of great interest. These transformations lead to invariant solutions also known as similarity solutions. Lie symmetry method provides a powerful tool for generation of these transformations [1, 2]. Consequently, the method enjoys a widespread application and has attracted attention of many researchers working in fields such as general relativity, nonlinear wave and diffusion equations [3-15].

For many nonlinear systems, there are only explicit exact solutions available. These solutions play an important role in both mathematical analysis and physical applications of the systems. A lot of research is being done in classification of symmetries [4, 8, 12, 13, 16-19], linearizing transformations and invariant solutions [9, 14, 20-28]. Symmetry analysis of a variety of 1-dimensional nonlinear wave equations has been given by various authors [17, 19, 23, 26, 29-32]. Studies have also been made for the 2-dimensional wave equation with constant coefficients [33, 34]. In this paper we provide a classification of a nonlinear 2-dimensional wave equation in which the non-linearity is introduced through a function representing the wave speed. Using the Lie symmetry criteria, we provide all possible symmetries by this wave equation and then use each of the sub-algebras of the symmetry generators to reduce the wave equation to an ordinary differential equation.

2. – Derivation of symmetry generators

In this section we derive symmetry generators of the 2-dimensional nonlinear wave equation,

$$u_{tt} = u^n(u_{xx} + u_{yy}). \tag{1}$$

In order to find the solution of (1), we use the Lie symmetry criterion for PDEs, $V^2(H)\big|_{H=0} = 0$, which leads to the expression

$$V^{(2)}\{u_{tt} - u^n(u_{xx} + u_{yy})\}|_{u_{tt}-u^n(u_{xx}+u_{yy})=0} = 0\,, \tag{2}$$

where $V^{(2)}$ represents the second prolongation of the generator associated with the original basis and is given by

$$\begin{aligned} V^{(2)} = V + \phi^x \frac{\partial}{\partial u_x} + \phi^y \frac{\partial}{\partial u_y} + \phi^t \frac{\partial}{\partial u_t} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xy} \frac{\partial}{\partial u_{xy}} + \\ + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{yy} \frac{\partial}{\partial u_{yy}} + \phi^{yt} \frac{\partial}{\partial u_{yt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}}\,. \end{aligned} \tag{3}$$

In the light of (3), eq. (2) becomes

$$\phi^{tt} - nu^{(n-1)}(u_{xx} + u_{yy})\phi - u^n(\phi^{xx} + \phi^{yy}) = 0. \tag{4}$$

In order to find coefficients of infinitesimal symmetry generator, we substitute the expressions for ϕ^j in (4). These expressions can be determined from the formula $\phi^j(x^i, u) = D_j\Big(\phi - \sum_{i=1}^3 \chi^i u_i\Big) + \sum_{i=1}^3 \chi^i u_{j,i}$ [1, 2]. Using these expressions in (4) gives rise to a determining system for the coefficients of infinitesimal symmetry generator. Comparing like terms in the resulting expression, we obtain a coupled system of equations. Using the coefficients of $u_{xx}u_x$, $u_{xx}u_{yt}$ and $u_{yy}u_{yt}$ in the determining system yields

$$\tau_u = \xi_u = \eta_u = 0. \tag{5}$$

In the light of (5) it is easy to find that the coefficients of the other terms in the derivatives of u give

$$\begin{cases} 2(\xi_x - \tau_t) = nu^{-1}\phi, \\ 2(\eta_y - \tau_t) = nu^{-1}\phi, \\ \eta_x = -\xi_y, \\ \xi_t = u^n\tau_x, \\ \eta_t = u^n\tau_y, \\ \phi_{uu} = 0. \end{cases} \tag{6}$$

Solving ϕ_{uu} from (6), immediately gives

$$\phi = \alpha(x,y,t)u + \beta(x,y,t). \tag{7}$$

Now substituting (7) in the remaining equations and solving them iteratively we obtain

$$\begin{cases} \xi = a_0 - a_1 y + a_2 x\,, \\ \eta = a_3 + a_1 x + a_2 y\,, \\ \tau = \Big(\dfrac{4a_2}{n+4} - \dfrac{2na_4}{n+4}\Big)t + a_5\,, \\ \phi = \Big(\dfrac{2a_2}{n+4} + \dfrac{4a_4}{n+4}\Big)u + \beta(x,y,t), \end{cases} \tag{8}$$

where all of $a_i's$ are arbitrary constants, giving a five-parameter group, G_5, with an additional parameter as an arbitrary function in the ϕ direction. From (8) we can construct the symmetry generators in explicit form as given below:

$$\begin{cases} V_0 = \dfrac{\partial}{\partial x}\,, \qquad V_1 = -y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}\,, \\ V_2 = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} + \Big(\dfrac{4t}{n+4}\Big)\dfrac{\partial}{\partial t} + \Big(\dfrac{2u}{n+4}\Big)\dfrac{\partial}{\partial u}\,, \\ V_3 = \dfrac{\partial}{\partial y}\,, \qquad V_4 = -\Big(\dfrac{2nt}{n+4}\Big)\dfrac{\partial}{\partial t} + \Big(\dfrac{4u}{n+4}\Big)\dfrac{\partial}{\partial u}\,, \\ V_5 = \dfrac{\partial}{\partial t}\,, \qquad V_\beta = \beta\dfrac{\partial}{\partial u}. \end{cases} \tag{9}$$

The symmetries V_0, V_3 and V_5 represent translations, V_1 rotation and V_β a scaling transformation. The arbitrary function $\beta(x,y,t)$ satisfies the wave equation in itself, given by

$$\beta_{tt} - u^n(\beta_{xx} + \beta_{yy}) = 0.$$

3. – Reductions

In this section we give reduction of (1) under two 2-dimensional subalgebras only: one when generators commute and second when they do not. A complete table of reductions by every single generator is given in appendix A and reductions under 2-dimensional subalgebras is given in appendix B. Further, before proceeding to show reductions in two cases, we construct commutation relations satisfied by each of the generators and give it in table I. According to Lie's theorem [1], if a PDE is invariant under V_i and V_j, it is also invariant under $[V_i, V_j]$ if it forms a closed sub-Lie algebra. With this in mind, we first consider the two mutually commuting generators V_3 and V_5 ($[V_3, V_5] = 0$) and give reduction of of the 2-dimensional wave equation under them.

TABLE I. – *Commutator algebra for symmetry generators.*

$[V_i, V_j]$	V_0	V_1	V_2	V_3	V_4	V_5
V_0	0	V_3	V_0	0	0	0
V_1	$-V_3$	0	0	V_0	0	0
V_2	$-V_0$	0	0	$-V_3$	0	$\left(\frac{-4}{n+4}\right)V_5$
V_3	0	$-V_0$	V_3	0	0	0
V_4	0	0	0	0	0	$\left(\frac{-2n}{n+4}\right)V_5$
V_5	0	0	0	$\left(\frac{4}{n+4}\right)V_5$	$\left(\frac{-2n}{n+4}\right)V_5$	0

Starting with $V_3 = \partial/\partial y$ we can reduce eq. (1) to

$$w_{rr} = w^n w_{ss} \tag{10}$$

with similarity variables $s = x,\ r = t$ and $w(r, s) = u$. Before using V_5 we first transform it to new variables given by

$$\widetilde{V}_5 = 0\frac{\partial}{\partial s} + \frac{\partial}{\partial r} + 0\frac{\partial}{\partial w}\,.$$

Solving the above equation it is straightforward to notice that the similarity variables for it become $\alpha = s$ and $\beta(\alpha) = w$. Using these variables reduces (10) to $\beta'' = 0$.

As a second example we consider generators V_0 and V_2. Whereas these generators do not commute with each other, their algebra is closed. We start reduction of (1) with V_0 and obtain

$$w_{rr} = w^n w_{ss}\,, \tag{11}$$

where $s = y,\ r = t$ and $w(r, s) = u$. Under these similarity variables, the V_2 transforms to $\widetilde{V}_2 = s\frac{\partial}{\partial s} + \left(\frac{4r}{n+4}\right)\frac{\partial}{\partial r} + \left(\frac{2w}{n+4}\right)\frac{\partial}{\partial w}$ with its invariants being $\alpha = r^{\frac{n}{4}+1}/s$ and $\sqrt{r}e^{\beta(\alpha)} = w$. Using these invariants (11) reduces to an ODE given by

$$-\frac{1}{4} + \left(\frac{n}{4}+1\right)^2\left(\alpha\beta' + \alpha^2\beta'^2 + \alpha^2\beta''\right) = \alpha^2 e^{n\beta}(2\alpha\beta' + \alpha^2\beta'^2 + \alpha^2\beta'').$$

A complete reduction under remaining two-dimensional subalgebras is given in Appendix B.

4. – Conclusions

Using Lie symmetry methods we give a set of symmetries of the non-linear wave equation in which the non-linearity has been introduced through u^n. Finding all the Lie point symmetries possessed by this equation we give a complete reduction of the wave equation to second-order ones.

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Appendix A.

Table II. – *Reduction under symmetry generators.*

Generator	Reduction and similarity variables
$V_0 = \frac{\partial}{\partial x}$	$w_{rr} = w^n w_{ss}$ where $s = y,\ r = t$ and $w(r,s) = u$
$V_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$	$w_{rr} = 4w^n(w_s + sw_{ss})$ where $s = x^2 + y^2,\ r = t$ and $w(r,s) = u$
$V_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} +$ $\left(\frac{4t}{n+4}\right)\frac{\partial}{\partial t} + \left(\frac{2u}{n+4}\right)\frac{\partial}{\partial u}$	$-\frac{1}{4} + \left(\frac{n}{4}+1\right)^2 r^2 w_r^2 + \left(\frac{n}{4}+1\right)w_r + \left(\frac{n}{4}+1\right)^2 r^2 w_{rr}$ $= r^2 e^{nw}\{r^2 w_r^2 + (s^2+1)w_s^2 + 2rsw_s w_r +$ $r^2 w_{rr} + 2srw_{sr} + 2rw_r + (s^2+1)w_{ss} + 2sw_s$ where $s = \frac{y}{x},\ r = \frac{t^{\frac{n}{4}+1}}{x}$ and $\sqrt{t}w^{w(r,s)} = u$
$V_3 = \frac{\partial}{\partial y}$	$w_{rr} = w^n w_{ss}$ where $s = x,\ r = t$ and $w(r,s) = u$
$V_4 = -\left(\frac{2nt}{n+4}\right)\frac{\partial}{\partial t} + \left(\frac{4u}{n+4}\right)\frac{\partial}{\partial u}$	$\frac{2}{n}\left(\frac{2}{n}+1\right) = e^{nw}(w_{ss} + w_{rr})$ where $s = x,\ r = y$ and $u = t^{-\frac{2}{n}}e^{w(r,s)}$
$V_5 = \frac{\partial}{\partial t}$	$w_{ss} + w_{rr} = 0$ where $s = x,\ r = y$ and $w(r,s) = u$

Appendix B.

Table III. – *Reductions under two-dimensional algebra.*

Algebra	Reductions and similarity variables
$[V_0, V_3] = 0$	$\beta'' = 0$ where $\alpha = r$, $\beta(\alpha) = w$ and $s = y,\ r = t,\ w(r,s) = u$
$[V_0, V_4] = 0$	$\frac{2}{n}\left(\frac{2}{n}+1\right) = e^{n\beta}(\beta'^2 + \beta'')$ where $\alpha = s,\ e^{\beta(\alpha)} r^{-\frac{2}{n}} = w$ and $s = y,\ r = t,\ w(r,s) = u$
$[V_0, V_5] = 0$	$\beta'' = 0$ where $\alpha = s,\ \beta(\alpha) = w$ and $s = y,\ r = t,\ w(r,s) = u$
$[V_1, V_2] = 0$	$-\frac{1}{2} + \left(\frac{n}{2}+2\right)^2(\alpha\beta' + \alpha^2\beta' + \alpha^2\beta'') = 4\alpha e^{n\beta}(\alpha\beta' + \alpha^2\beta'^2 + \alpha^2\beta'')$ where $\alpha = \frac{r^{\frac{n}{2}+2}}{s},\ \sqrt{r}e^{\beta(\alpha)} = w$ and $s = x^2 + y^2,\ r = t,\ w(r,s) = u$
$[V_1, V_4] = 0$	$\frac{2}{n}\left(\frac{2}{n}+1\right) = 4e^{n\beta}\{\beta' + \alpha(\beta'^2 + \beta'')\}$ where $\alpha = s,\ e^{\beta(\alpha)} r^{-\frac{2}{n}} = w$ and $s = x^2 + y^2,\ r = t,\ w(r,s) = u$
$[V_1, V_5] = 0$	$\beta' + \alpha\beta'' = 0$ where $\alpha = s^2 + r^2,\ \beta(\alpha) = w$ and $s = x,\ r = y,\ w(r,s) = u$
$[V_2, V_4] = 0$	$\frac{2}{n}\left(\frac{2}{n}+1\right) = e^{n\beta}\left(\beta'' - \frac{2}{n} + \alpha^2\beta'' + 2\alpha\beta'\right)$ where $\alpha = \frac{s}{r},\ \frac{2}{n}\ln r + \beta(\alpha) = w$ and $s = x,\ r = y,\ w(r,s) = \ln\left(ut^{\frac{2}{n}}\right)$

TABLE III. – *Continued.*

Algebra	Reductions and similarity variables
$[V_3, V_4] = 0$	$\frac{2}{n}\left(\frac{2}{n}+1\right) = e^{n\beta}(\beta'' + \beta'^2)$ where $\alpha = s,\ e^{\beta(\alpha)}r^{-\frac{2}{n}} = w$ and $s = x,\ r = t,\ w(r,s) = u$
$[V_3, V_5] = 0$	$\beta'' = 0$ where $\alpha = s,\ \beta(\alpha) = w$ and $s = x,\ r = t,\ w(r,s) = u$
$[V_0, V_2] = V_0$	$-\frac{1}{4} + \left(\frac{n}{4}+1\right)^2\left(\alpha\beta' + \alpha^2\beta'^2 + \alpha^2\beta''\right) = \alpha^2 e^{n\beta}(2\alpha\beta' + \alpha^2\beta'^2 + \alpha^2\beta'')$ where $\alpha = \frac{r^{\frac{n}{4}+1}}{s},\ \sqrt{r}e^{\beta(\alpha)} = w$ and $s = y,\ r = t,\ w(r,s) = u$
$[V_3, V_2] = V_3$	$-\frac{1}{4} + \left(\frac{n}{4}+1\right)^2\left(\alpha\beta' + \alpha^2\beta'^2 + \alpha^2\beta''\right) = \alpha^2 e^{n\beta}(2\alpha\beta' + \alpha^2\beta'^2 + \alpha^2\beta'')$ where $\alpha = \frac{r^{\frac{n}{4}+1}}{s},\ \sqrt{r}e^{\beta(\alpha)} = w$ and $s = x,\ r = t,\ w(r,s) = u$
$[V_5, V_2] = c_1 V_5$	$\frac{2(n+2)}{(n+4)^2} = \left(\frac{2n+12}{n+4}\right)\alpha\beta' + \beta'^2 + \beta'' + \alpha^2\beta'^2 + \alpha^2\beta''$ where $\alpha = \frac{r}{s},\ s^{\frac{2}{n+4}}e^{\beta(\alpha)} = w$ and $s = x,\ r = y,\ w(r,s) = u$

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