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The structure of the pseudo-Newtonian force and potential about a five-dimensional rotating black hole

M. Sharif(*)

Department of Mathematics, University of the Punjab
Quaid-e-Azam Campus Lahore-54590, Pakistan

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Summary. — In this paper we investigate the structure of the pseudo-Newtonian force and potential about a five-dimensional rotating black hole. The conditions for the force character from an attractive to repulsive are considered. It is also found that the force will reach a maximum under certain conditions.

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Einstein's theory of relativity replaces the use of forces in dynamics by what Wheeler calls *geometrodynamics*. Paths are bent, not by forces, but by the *curvature of space-time*. In the process the guidance of the intuition based on the earlier dynamics is lost. However, our intuition continues to reside in the force concept, particularly when we have to include other forces in the discussion. For this reason one may want to reverse the procedure of general relativity and look at the non-linear *force of gravity* which would predict the same bending of the path as predicted by geometry. In the pseudo-Newtonian (ψN) approach [1, 2], the curvature of the space-time is *straightened out* to yield a relativistic force which bends the path, so as to again supply the guidance of the earlier, force-based, intuition.

The relativistic analogue of the Newtonian gravitational force which gives the relativistic expression for the tidal force in terms of the curvature tensor is called the ψN force. The quantity whose gradient gives the ψN force is the ψN potential. The ψN force is the generalisation of the force which gives the usual Newtonian force for the Schwarzschild metric and a Q^2/r^3 correction to it in the Riessner-Nordstrom metric. The ψN force may be regarded as the *Newtonian fiction* which *explains* the same motion (geodesic) as the *Einsteinian reality* of the curved space-time does. We can, thus, translate back to Newtonian terms and concepts where our intuition may be able to lead us to ask, and answer, questions that may not have occurred to us in relativistic terms. Some

(*) E-mail: msharif@math.pu.edu.pk

insights have already been obtained [1-4] by expressing the consequences of general relativity in terms of forces by applying it to Kerr and Kerr-Newmann metrics. Recently, the pseudo-Newtonian potential has been evaluated for charged particle in Kerr-Newmann geometry by Ivanov and Prodanov [5]. In this paper, we study the structure of force and potential about a five-dimensional rotating black hole.

The basis of the ψN formalism is the observation that, whereas the gravitational force is not detectable in a freely falling frame, *that is so only at a point.* It is detectable over a finite spatial extent as the tidal force. The tidal force, which is operationally determinable, can be related to the curvature tensor by

$$F_T^\mu = mR^\mu_{\nu\rho\pi}t^\nu l^\rho t^\pi \qquad (\mu,\nu,\rho,\pi = 0,1,2,3), \tag{1}$$

where m is the mass of a test particle, $t^\mu = f(x)\delta^\mu_0$, $f(x) = (g_{00})^{-1/2}$ and l^μ is the separation vector. l^μ can be determined by the requirement that the tidal force have maximum magnitude in the direction of the separation vector. Choosing a gauge in which $g_{0i} = 0$ (similar to the synchronous coordinate system [6,7]) in a coordinate basis. We further use Riemann normal coordinates for the spatial direction, but not for the temporal direction. The reason for this difference is that both ends of the accelerometer are spatially free, *i.e.* both move and do not stay attached to any spatial point. However, there is a *memory* of the initial time built into the accelerometer in that the zero position is fixed then. Any change is registered that way. Thus *time* behaves very differently from *space.*

The ψN force, F_μ, satisfies the equation

$$F_T^{*\mu} = l^\nu F^\mu_{;\nu}, \tag{2}$$

where $F_T^{*\mu}$ is the extremal tidal force corresponding to the maximum magnitude reading on the dial. Notice that $F_T^{*0} = 0$ does not imply that $F^0 = 0$. The requirement that eq. (2) be satisfied can be written as

$$l^i(F^0_{,i} + \Gamma^0_{ij}F^j) = 0, \tag{3}$$

$$l^j(F^i_{,j} + \Gamma^i_{0j}F^0) = F_T^{*i} \qquad (i,j = 1,2,3). \tag{4}$$

A simultaneous solution of the above equations can be found by taking the ansatz

$$F^0 = m\left[(\ln A)_{,0} - \Gamma^0_{00} + \Gamma^i_{0j}\Gamma^i_{0i}/A\right]f^2, \tag{5}$$

$$F^i = m\Gamma^i_{00}f^2, \tag{6}$$

where $A = (\ln\sqrt{-g})_{,0}$, $\quad g = \det(g_{ij})$. These equations can be written in terms of two quantities U and V given by

$$U = m\left[\ln(Af/B) - \int (g^{ij}_{,0}g_{ij,0}/4A)\mathrm{d}t\right], \tag{7}$$

$$V = -m\ln f, \tag{8}$$

as

$$F_0 = -U_{,0}, \quad F_i = -V_{,i}, \tag{9}$$

where B ia a constant with units of time inverse so as to make A/B dimensionless. It is to be noted that the momentum four-vector p_μ can be written in terms of the integral of the force four-vector F_μ. Thus

$$p_\mu = \int F_\mu \mathrm{d}\tau. \tag{10}$$

Notice that the zero component of the momentum four-vector corresponds to the energy imparted to a test particle of mass m while the spatial components give the momentum imparted to a test particle.

Thus, in the free fall rest-frame, the ψN force is given [2, 3] by

$$F_i = -m(\ln\sqrt{g_{00}})_{,i}, \quad = -V_{,i}, \tag{11}$$

where

$$V = m(\ln\sqrt{g_{00}}). \tag{12}$$

It is clear that V is the generalisation of the classical gravitational potential and, for small variations from Minkowski space

$$V \approx \frac{1}{2}m(g_{00} - 1) \tag{13}$$

which is the pseudo-Newtonian potential. We shall analyse the behaviour of these quantities for the five-dimensional rotating black hole.

The metric of a rotating black hole in five dimensions follows from the general asymptotically flat solutions to $(N+1)$-dimensional vacuum gravity found by Myers and Perry [8]. In Boyer-Lindquist–type coordinates, it takes the most simple form given by

$$\begin{aligned} \mathrm{d}s^2 = {} & -\mathrm{d}t^2 + \Sigma(\frac{r^2}{\Pi}\mathrm{d}r^2 + \mathrm{d}\theta^2) + (r^2+a^2)\sin^2\theta\mathrm{d}\phi^2 + (r^2+b^2)\cos^2\theta\mathrm{d}\psi^2 + \\ & + \frac{M}{\Sigma}(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi - b\cos^2\theta\mathrm{d}\psi)^2, \end{aligned} \tag{14}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta + b^2\sin^2\theta, \quad \Pi = (r^2+a^2)(r^2+b^2) - Mr^2, \tag{15}$$

and M is a parameter related to the physical mass of the black hole, while the parameters a and b are associated with its two independent angular momenta.

The event horizon of the black hole is a null surface determined by the equation

$$\Pi = (r^2+a^2)(r^2+b^2) - Mr^2 = 0. \tag{16}$$

The largest root of this equation gives the radius of the black hole's outer event horizon. We have

$$r_h^2 = \frac{1}{2}(M - a^2 - b^2 + \sqrt{(M - a^2 - b^2)^2 - 4a^2b^2}. \tag{17}$$

Notice that the horizon exists if and only if

$$a^2 + b^2 + 2|ab| \leq M, \tag{18}$$

so that the condition $M = a^2 + b^2 + 2|ab|$ or, equivalently, $r_{\rm h}^2 = |ab|$ defines the extremal horizon of a five-dimensional black hole.

In the absence of the black hole ($M = 0$), the metric (14) reduces to the flat one written in oblate bi-polar coordinates. On the other hand, for $a = b = 0$ we have the Schwarzschild-Tangherlini [9] static solution in spherical bi-polar coordinates. The bi-azimuthal symmetry properties of the five-dimensional black-hole metric (14) along with its stationarity imply the existence of the three commuting Killing vectors

$$\xi_{(t)} = \frac{\partial}{\partial t}, \quad \xi_{(\phi)} = \frac{\partial}{\partial \phi}, \quad \xi_{(\psi)} = \frac{\partial}{\partial \psi}. \tag{19}$$

The scalar products of these Killing vectors are expressed through the corresponding metric components as follows:

$$\begin{aligned}
\xi_{(t)}.\xi_{(t)} &= g_{tt} = -1 + \frac{M}{\Sigma}, \\
\xi_{(\phi)}.\xi_{(\phi)} &= g_{\phi\phi} = \left(r^2 + a^2 + \frac{Ma^2}{\Sigma}\sin^2\theta\right)\sin^2\theta, \\
\xi_{(t)}.\xi_{(\phi)} &= g_{t\phi} = -\frac{Ma}{\Sigma}\sin^2\theta, \\
\xi_{(\psi)}.\xi_{(\psi)} &= g_{\psi\psi} = \left(r^2 + b^2 + \frac{Mb^2}{\Sigma}\cos^2\theta\right)\cos^2\theta, \\
\xi_{(t)}.\xi_{(\psi)} &= g_{t\psi} = -\frac{Mb}{\Sigma}, \\
\xi_{(\phi)}.\xi_{(\psi)} &= g_{\phi\psi} = \frac{Mab}{\Sigma}\sin^2\theta\cos^2\theta.
\end{aligned} \tag{20}$$

The Killing vectors (19) can be used to give a physical interpretation of the parameters M, a and b involved in the metric (14). One can obtain coordinate-independent definitions for these parameters by using the analysis given in [10]. We have the integrals

$$M = \frac{1}{4\pi^2}\oint \xi_{(t)}^{\mu;\nu} {\rm d}^3\Sigma_{\mu\nu}, \tag{21}$$

$$j(a) = aM = -\frac{1}{4\pi^2}\oint \xi_{(\phi)}^{\mu;\nu} {\rm d}^3\Sigma_{\mu\nu}, \quad j(b) = bM = -\frac{1}{4\pi^2}\oint \xi_{(\psi)}^{\mu;\nu} {\rm d}^3\Sigma_{\mu\nu}, \tag{22}$$

where the integrals are taken over the 3-sphere at spatial infinity,

$$\mathrm{d}^3\Sigma_{\mu\nu} = \frac{1}{3!}\sqrt{-g}\varepsilon_{\mu\nu\alpha\beta\gamma}\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma, \tag{23}$$

the semicolon denotes covariant differentiation. The two specific angular momentum parameters $j(a)$ and $j(b)$ are associated with rotations in the ϕ and ψ directions, respectively. It is mentioned here that with these definitions the relation between the specific angular momentum and the mass parameter looks exactly like the corresponding relation ($J = aM_T$) of four-dimensional Kerr metric. It can be shown that the definitions given in (21) and (22) do in fact correctly describe the mass and angular momenta parameters. The integrands can be calculated in the asymptotic region $r \to \infty$. The dominant terms in the asymptotic expansion have the form

$$\begin{aligned} \xi^{t;r}_{(t)} &= \frac{M}{r^3} + O\left(\frac{1}{r^5}\right), \\ \xi^{t;r}_{(\phi)} &= -\frac{2aM\sin^2\theta}{r^3} + O\left(\frac{1}{r^5}\right), \\ \xi^{t;r}_{(\psi)} &= -\frac{2bM\cos^2\theta}{r^3} + O\left(\frac{1}{r^5}\right). \end{aligned} \tag{24}$$

One can easily verify that these expressions satisfy formulae (21) and (22). On the other hand, the relation of the above parameters to the total mass M_T and the total angular momenta $J(a)$ and $J(b)$ of the black hole can be established using the formulae given in [8]. We obtain that

$$M = \frac{8}{3\pi}M_T, \quad j(a) = \frac{4}{\pi}J(a), \quad j(b) = \frac{4}{\pi}J(b)\,. \tag{25}$$

These relations confirm the interpretation of the parameters M, a and b as being related to the physical mass and angular momenta of the metric (14).

The structure of the ψN force (per unit mass of the test particle) for the five-dimensional rotating black hole turns out to be the following:

$$F_r = -\frac{Mr}{\Sigma(\Sigma - M)}\,, \tag{26}$$

$$F_\theta = \frac{M(a^2 - b^2)\sin\theta\cos\theta}{\Sigma(\Sigma - M)}\,. \tag{27}$$

We see that the radial component can never become zero outside the horizon and so the force cannot change character from an attractive to a repulsive force outside the black hole. The polar component can only become zero outside the horizon at $\theta = 0,\ \pi/2,\ \pi$. We note that naked singularities can give repulsive as well as attractive forces. The force structure can provide interesting features if it reaches a maximum and then drops as we reduce r or change θ provided the turnover lies outside the horizon. Since our observers are seeing *force* in a flat space, the metric to be used is the plane polar one. Thus the square of the magnitude of the force is

$$(F)^2 = \frac{M^2}{r^2\Sigma^2(\Sigma - M)^2}[r^4 + (a^2 - b^2)^2 \sin^2\theta \cos^2\theta]. \tag{28}$$

It follows from here that

$$|F| = \frac{M}{r\Sigma(\Sigma - M)}[r^4 + (a^2 - b^2)^2 \sin^2\theta \cos^2\theta]^{1/2}. \tag{29}$$

The expansion of $|F|$ in powers of $(1/r)$ can be given as

$$|F| = \frac{Mr}{\Sigma(\Sigma - M)}\left[1 + \frac{(a^2 - b^2)^2 \sin^2\theta \cos^2\theta}{2r^4} + ...\right]. \tag{30}$$

This shows that the force far from the center, *i.e.* when r is very very large, becomes zero. This corresponds to the behaviour of the Kerr metric [4].

The equations for the turnovers along r and θ, respectively, are

$$\begin{aligned} &r^4[\Sigma(\Sigma - M) - 2r^2(2\Sigma - M)] - (a^2 - b^2)^2 \cdot \\ &\cdot[\Sigma(\Sigma - M) + 2r^2(2\Sigma - M)] \sin^2\theta \cos^2\theta = 0, \end{aligned} \tag{31}$$

$$\Sigma(\Sigma - M)(a^2 - b^2) \cos 2\theta + 2(2\Sigma - M)[r^4 + (a^2 - b^2)^2 \sin^2\theta \cos^2\theta] = 0. \tag{32}$$

Equations (31) and (32) are not easy to analyse generally. We investigate these equations for the following two special cases.

i) $a = b \neq 0$, ii) $a = 0 = b$

For the first case i), when the two angular momenta are the same, eqs. (31) and (32) reduce to

$$(r^2 + a^2)(r^2 + a^2 - M) - 2r^2(2r^2 + 2a^2 - M) = 0, \tag{33}$$

$$2(r^2 + a^2) - M = 0. \tag{34}$$

It follows that the first equation is satisfied when

$$r^2 = \frac{1}{6}[(M - 2a^2) \pm \sqrt{M^2 + 16a^4 - 16Ma^2}]. \tag{35}$$

Equation (34) is satisfied for the value of r given by

$$r^2 = \frac{1}{2}M - a^2. \tag{36}$$

If we take $a = 0 = b$, *i.e.* there is no rotation, the values of r satisfying eqs. (31) and (32), respectively, are

$$r = \pm\sqrt{\frac{M}{3}}, \quad r = \pm\sqrt{\frac{M}{2}}. \tag{37}$$

We note that in the first case, a maximum of the magnitude does occur at the value of r given by eq. (35). The maximum value of the force can be obtained by replacing this value of r in F_r. Similarly, for the case ii), a maximum value of the magnitude is obtained for r as given in eq. (37).

The corresponding potential is obtained by using eqs. (13) and (14) and is given by

$$V = -\frac{m}{2\Sigma}. \tag{38}$$

For the special case i), it becomes

$$V = -\frac{m}{2(r^2 + a^2)}. \tag{39}$$

When there is no rotation it reduces to

$$V = -\frac{m}{2r^2}. \tag{40}$$

We note that the structure of force and potential indicate similar type of behaviour as for the Kerr metric [4].

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