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On q-deformed Pöschl-Teller potentials(*)

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Summary. — A simple algebraic technique is developed to obtain deformed energy spectra for the Pöschl-Teller potentials.

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1. – Introduction

There has been an increasing interest in quantum deformed systems during the past decade [1]. The use of q-deformed algebras has been seen as a possible generalization of the usual algebrization of the Schrödinger equation with the use of Lie algebras. There are several approaches to this problem. Bonatsos *et al.* [2] have studied deformed harmonic and anharmonic oscillators as a possible description of vibrational spectra of diatomic molecules. More recently, Cooper *et al.* [3] have studied the deformation of the Morse potential using supersymmetric quantum mechanics (SSQM) and a group approach. It is interesting to notice that almost all physical problems for which a deformation has been carried out belong to the class of potentials related to confluent hypergeometric functions, *i.e.* Coulomb, harmonic oscillator and Morse potentials, the only exception seems the case treated in [4].

In this paper, following the ideas developed in [3], we deal with two potentials whose solutions are hypergeometric functions, the Pöschl-Teller I and II, using only the spectrum generating algebra associated with these potentials. No reference to SSQM is used.

Recently in [5] it was shown that the hypergeometric Natanzon potentials [6], V_N—those for which the Schrödinger equation can be transformed to a hypergeometric function—can be solved algebraically by means of the $SO(2,1)$ algebra. This is an essential

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point for what is going to be done later, so it is convenient to make a brief review of the subject. The basic assumptions of this approach are: a) a two-variable realization of $SO(2,1)$. b) The Schrödinger equation can be written in terms of the Casimir operator of the algebra C, as $[H-E]\,\Psi(r,y) = G(r)[C-c]\Psi(r,y)$, where c is the eigenvalue of C , H the Hamiltonian and E the corresponding eigenvalue. $G(r)$ is a function fixed by consistency, and c) The eigenfunctions of the Hamiltonian have the form $\Psi(r,y) = \exp[imy]\Phi(r)$.

The hypergeometric Natanzon potentials are given by

$$V_N = \frac{1}{R}(f\ z(r)^2 - (\ h_0 - h_1 + f\ )\ z(r) + h_0 + 1) + \\ + \frac{z(r)^2(1-z(r))^2}{R^2}\left[a + \frac{a + (c_1 - c_0)(2\ z(r) - 1)}{z(r)\ (z(r)-1)} - \frac{5\Delta}{4R}\right], \tag{1}$$

where

$$\Delta = \tau^2 - 4ac_0\ ,\ \tau = c_1 - c_0 - a\ ,\ R = a\ z(r)^2 + \tau\ z(r) + c_0\ . \tag{2}$$

The constants a, c_0, c_1, h_0, h_1 and f are called Natanzon parameters. The function $z(r)$ must satisfy

$$\frac{\mathrm{d}z(r)}{\mathrm{d}r} = \frac{2z(r)(1-z(r))}{\sqrt{R}} \tag{3}$$

We follow the notation of [7].

The generators of the $SO(2,1)$ algebra: J_1, J_2 and J_0 satisfy the usual commutation relations: $[J_0, J_1] = iJ_2$, $[J_2, J_0] = iJ_1$, $[J_1, J_2] = -iJ_0$, as usual we define $J_\pm = J_1 \pm i\,J_2$. The Casimir operator C is given by $C = J_0(J_0 \pm 1) - J_\mp J_\pm$ .The generators are then given by

$$\exp[\mp iy]\,J_\pm = \pm\left(\frac{z(r)^{1/2}(z(r)-1)}{z(r)'}\right)\frac{\partial}{\partial r} - \left(\frac{i}{2}\frac{(z(r)+1)}{\sqrt{z(r)}}\right)\frac{\partial}{\partial y} + \\ + \frac{(z(r)-1)}{2}\left[\frac{(p\mp 1)}{\sqrt{z(r)}} - \frac{\sqrt{z(r)}\,z(r)''}{z(r)'^{\,2}}\right], \tag{4}$$

$$J_0 = -i\frac{\partial}{\partial y}\,, \tag{5}$$

where $z(r)' = \mathrm{d}z(r)/\mathrm{d}r$ and p is a function of the Natanzon parameters and they generally depend on the energy of the system. The Casimir operator turns out to be

$$C = (z(r)-1)^2\left[\frac{z(r)}{z(r)'^{\,2}}\frac{\partial^2}{\partial r^2} + \frac{i}{4\sqrt{z(r)}}\frac{\partial^2}{\partial y^2} + \frac{i\ p\ (z(r)+1)}{2\ (z(r)-1)\ z(r)}\frac{\partial}{\partial y}\right] + \tag{6}$$

$$+(z(r)-1)^2 \left[\frac{z(r)\; z(r)'''}{2\; z(r)'^{\;2}} - \frac{3\; z(r)\; z(r)''^{\;2}}{4\; z(r)'^{\;4}} - \frac{(p^2-1)}{4\; z(r)} \right] .$$

The eigenvalues of the compact generator, (5), are known to be

$$m = \nu + \frac{1}{2} + \sqrt{c + \frac{1}{4}}\,, \qquad \nu = 0, 1, ... \tag{7}$$

and the energy spectra are given by

$$2\nu + 1 = \alpha(\nu) - \beta(\nu) - \delta(\nu)\,, \tag{8}$$

where

$$\begin{aligned} \alpha(\nu) &= \sqrt{-aE(\nu) + f + 1}\,, \\ \beta(\nu) &= \sqrt{-c_0 E(\nu) + h_0 + 1}\,, \\ \delta(\nu) &= \sqrt{-c_1 E(\nu) + h_1 + 1}\,. \end{aligned} \tag{9}$$

The last relevant relation for this work is the connection between the eigenvalues of the Casimir operator, c , with $E(\nu)$ and the Natanzon parameters:

$$\sqrt{-c_1 E(\nu) + h_1 + 1} = \sqrt{4\; c + 1}\,. \tag{10}$$

2. – Deformed potentials

We proceed in a similar way as in ref. [3] to obtain the deformed energy spectra of the Pöschl-Teller potentials. For this purpose we consider the following bosonic realization of $SO(2,1)$ [8]:

$$J_+ = a^\dagger b^\dagger \quad , \; J_- = ab \quad , \; J_0 = \frac{1}{2}\; \left(a^\dagger a + b^\dagger b + 1\right)\,, \tag{11}$$

the operators $a, a^\dagger, b$ and $b^\dagger$ have the usual commutation relations : $\left[a, a^\dagger\right] = \left[b, b^\dagger\right] = 1$, while the other commutators are zero. The basis for this realization is given by

$$|j\; m\rangle = \frac{a^{\dagger m-j} b^{\dagger m+j-1}}{\sqrt{(m-j)!\;(m+j-1)!}}\; |0\rangle \tag{12}$$

in this basis, the operators defined in (11) satisfy the following relations:

$$J_\pm\; |j\; m\rangle = \sqrt{(m \mp j \pm 1)\;(m \pm j)}\; |j\; m \pm 1\rangle\,, \tag{13}$$

$$J_0 \; |j \; m\rangle = m \; |j \; m\rangle \, . \tag{14}$$

The Casimir operator has eigenvalues: $j\ (j-1)$ and j is assumed to be positive, m takes the values: $m = j + \nu$, $\nu = 0, 1...$

The deformed states are defined as [9]

$$|j \; m\rangle_q = \frac{a^{\dagger m-j} b^{\dagger m+j-1}}{\sqrt{[m-j]! \; [m+j-1]!}} \; |0\rangle_q \, , \tag{15}$$

where $[n]! = [1][2]..., [n]$, with

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{16}$$

and the q-boson operators satisfy the following commutations relations:

$$aa^\dagger - qa^\dagger a = q^{-N_a} \; , \; [N_a \; , a^\dagger] = a^\dagger \; , \; [N_a \; , a] = -a \tag{17}$$

with similar relations for the b operators. The relations given in (13) are transformed into the following ones:

$$J_\pm \; |j \; m\rangle_q = \sqrt{[m \pm j] \; [m \mp j \pm 1]} \;\; |j \; m \pm 1\rangle_q \, , \tag{18}$$

$$J_0 \; |j \; m\rangle_q = m \; |j \; m\rangle_q \, . \tag{19}$$

Then the ladder operators defined in (13) satisfy the $SO(2,1)_q$ algebra

$$[J_0, \; J_\pm] = \pm \; J_0 \; , \quad [J_+, \; J_-] = -[2 \; J_0] \, . \tag{20}$$

We can now find the deformed energy spectrum in the following way: First we build up a bosonic representation of the Hamiltonian for the system in consideration—$H_{\rm b}$—in such a way that its spectrum coincides with the one obtained from (8). The next step is a deformation of $H_{\rm b}$ and the calculation of its spectrum. Let us consider a few examples, we follow the notation of [10].

1) *Pöschl-Teller I potential*

This potential is defined by

$$V = -(A+B)^2 + A(A-\alpha)\sec^2(\alpha x) + B(B-\alpha)\csc^2(\alpha x)\, ; \tag{21}$$

it is easily seen that the Natanzon parameters for this potential are given by

$$a = 0, \;\; c_0 = 0, \;\; c_1 = -\frac{1}{\alpha^2}, \;\; f = \frac{(2A+\alpha)\;(2A-3\alpha)}{4\alpha^2}\,, \tag{22}$$
$$h_0 = \frac{(2B+\alpha)\;(2B-3\alpha)}{4\alpha^2}, \;\; h_1 = \frac{(A+B+\alpha)\;(A+B-\alpha)}{\alpha^2}\,;$$

the function $z(x)$ is obtained from (3) and it becomes $z(x) = -\tan^2(\alpha x)$. After a careful study of the signs of the square roots occurring in (8) one finds that the energy spectrum of this system is

$$E(\nu) = 4\upsilon\alpha(\upsilon\alpha + A + B) \quad , \quad \nu = 0, 1, \dots . \tag{23}$$

From (10) we obtain for c, the eigenvalue of the Casimir operator,

$$c = \frac{1}{4\alpha^2}\left((A+B+2\nu\alpha)^2 - \alpha^2\right)\,; \tag{24}$$

from (7) m is found to be

$$m = 2\nu + \frac{1}{2\alpha}(A+B+\alpha) = 2\nu + \lambda \tag{25}$$

and from (24)

$$j = \nu + \frac{1}{2\alpha}(A+B+\alpha) = \nu + \lambda\,, \tag{26}$$

where the parameter λ is defined as: $\lambda = \frac{1}{2\alpha}(A+B+\alpha)$. Denoting the adimensional energy of the system by $e(\nu)$, we have from (23)

$$e(\nu, \lambda) = 4\nu(\nu + 2\lambda - 1)\,. \tag{27}$$

For the values of j and m given in (25), (26), the corresponding state will be written as $|\nu\;\lambda\rangle$, then it is easily seen that: $a^\dagger a\;|\nu\;\lambda\rangle = \nu\;|\nu\;\lambda\rangle$ and $b^\dagger b\;|\nu\;\lambda\rangle = (2\lambda+\nu-1)\;|\nu\;\lambda\rangle$. With these results we can build a bosonic representation of the Hamiltonian operator H_{pt1} for this system as follows:

$$H_{\mathrm{pt1}} = a^\dagger a\;(b^\dagger b - 2a^\dagger a)\,, \tag{28}$$

then we have the expected result

$$H_{\mathrm{pt1}}\;|\nu\;\lambda\rangle = e(\nu,\lambda)\;|\nu\;\lambda\rangle\,. \tag{29}$$

Keeping the same form for the deformed Hamiltonian as the one given in (28), we obtain for the deformed version of (29)

$$H_{\text{dpt1}}\ |\nu\ \lambda\rangle_q = e(\nu,\lambda)_q\ |\nu\ \lambda\rangle_q \tag{30}$$

with

$$e(\nu,\lambda)_q = 4\ [\nu]\ ([2\lambda+3\nu-1]-2[\nu]) \tag{31}$$

and $H_{\text{dpt1}} \equiv a_q^\dagger a_q(b_q^\dagger b_q - 2a_q^\dagger a_q)$. The expression (31) can be considered as the deformed energy spectrum of the system, its limit as $q \to 1$ agrees with the undeformed energy given in (23).

2) *Pöschl-Teller II potential*
It is defined by

$$V = (A-B)^2 - A(A+\alpha)\text{sech}^2(\alpha r) + B(B-\alpha)\text{csch}^2(\alpha r)\,. \tag{32}$$

The Natanzon parameters are

$$\begin{gathered} a = 0,\ \ c_0 = 0,\ \ c_1 = \frac{1}{\alpha^2},\ \ f = \frac{(2A-\alpha)(2A+3\alpha)}{4\alpha^2}\,, \\ h_0 = \frac{(2B+\alpha)(2B-3\alpha)}{4\alpha^2},\ \ h_1 = \frac{(A-B+\alpha)(A-B+\alpha)}{\alpha^2}\,. \end{gathered} \tag{33}$$

With this set of parameters one obtains from (3), (8), (9) , $z(r) = \tanh^2(\alpha r)$ and for the energy spectra

$$E(\nu) = -4\nu\alpha\ (\nu\alpha - A + B)\quad ,\quad \nu = 0,1,\ldots\,. \tag{34}$$

In this case c, m and j are given by

$$\begin{aligned} c &= \frac{1}{4\alpha^2}\ \left((A-B-2\nu\alpha)^2-\alpha^2\right)\,, \\ m &= \frac{1}{2\alpha}\ (A-B+\alpha) \equiv \lambda\,, \\ j &= \frac{1}{2\alpha}\ (A-B+\alpha) - \nu = \lambda - \nu\,, \end{aligned} \tag{35}$$

while $e(\nu,\lambda)$ is now

$$e(\nu,\lambda) = 4\nu\ (2\lambda-\nu-1)\,. \tag{36}$$

The corresponding Hamiltonian operator is a simple one

$$H_{\text{pt2}} = 4\ J_+J_-\,. \tag{37}$$

For the deformed energy spectrum we find

$$e(\nu,\lambda)_q = 4\ [\nu]\ [2\lambda-\nu-1]\,. \tag{38}$$

One easily see that the limiting case of (38) agrees with the result given in (36).

At this point we would like to make a few remarks. One may ask why we are considering these two potentials and not analyzing the rest of the shape-invariant hypergeometric potentials listed for example in [10]. There are several reasons. We first notice that the energy spectra of these class of potentials have two main features, one is that there is a sub-class of potentials whose energy spectra is a quadratic expression involving ν while the spectrum of the other ones is a ratio of quadratic polynomials in ν. In the first case, there are only two cases where two Natanzon parameters occur in the energy spectra, these are the cases treated here, in the remaining two cases only one parameter appears. For the latter cases one can assume that one parameter is proportional to the other and the results are not so simple as the cases treated in this paper. For the remaining set of potentials, the bosonization is not so clear since one has to get ratios of quadratics polynomials in ν .

Finally we would like to mention that a polynomial deformation, in the sense of Delbecq *et al.* [11], can be done in an simple way for the Pöschl-Teller II potential. This is so because of the simple structure of H_{pt2} given in (37). The algebra that we use is the $\mathcal{A}_q^-\ (2,1)$, and we use the $D_q^{(+)}$ representation which is bounded below. These algebra satisfy the following commutation relations:

$$\begin{aligned}
[J_0\ ,\ J_+] &= (1+(1-q)\ J_0)\ J_+\,,\\
[J_0\ ,\ J_+] &= -J_-\ (1+(1-q)\ J_0)\,,\\
[J_+\ ,\ J_-] &= -2\ J_0\ (1+(1-q)\ J_0)\,.
\end{aligned} \tag{39}$$

For this algebra the Casimir operator is given by

$$C_q = J_+\ J_- - \frac{2}{1+q}\ (q\ J_0-1)\ J_0\,. \tag{40}$$

The eigenvalues of the Casimir operator are written as follows:

$$c_q = -\frac{2}{(1+q)}\phi\ (\phi+1) \tag{41}$$

with $\phi = q\ j < 0$. The eigenvalues of J_0 are

$$m_q = -j\ q^{-\upsilon} - \frac{1-q^{-\upsilon}}{1-q}, \quad \nu = 0,1,\ldots\,. \tag{42}$$

Notice that $m_q \to m$ in the limit $q \to 1$, in other words $D_q^{(+)} \to D^{(+)}$ which is the representation that is used in the standard algebraic description of the Natanzon potentials.

As before, the deformed Hamiltonian is assumed to have the same form given in (37), then from (40) we can write this deformation as

$$H'_{\mathrm{pt2}} = 4\ (C_q + \frac{2}{1+q}\ (q\ J_0 - 1)\ J_0)\,. \tag{43}$$

The eigenvalues of H'_{pt2} are thus given by

$$e'(\nu,\lambda)_q = \frac{8}{1+q}\ (\ m_q\ (qm_q - 1) - qj\ (qj+1)\ )\,, \tag{44}$$

where in the expression for m_q we must use the results given in (35), namely $m = \lambda$ and $j = \lambda - \nu$ .This is the deformed energy spectra for deformation in consideration. It is easy to prove that in the $q \to 1$ limit one recovers the result given in (36).

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