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General relativistic forces in non-static spacetimes (*)

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Summary. — The pseudo-Newtonian (ψN) force was defined as the General Relativistic analogue of the Newtonian gravitational force for static spacetimes. It corresponds to the force in a Fermi-normal frame. The extended ψN(eψN) force for non-static spacetimes gives the ψN force in the static spacetime and an expression for the force in some non-static spacetimes. In particular, it gives an exact formula for the momentum imparted to test particles by gravitational waves. However, the frame used for the eψN force was not identified. Here we review the developments of the ψN and eψN formalism and some recent work reducing the apparent arbitrariness in the choice of frame. Some open problems are identified.

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1. – Introduction

General Relativity (GR) replaces the Newtonian gravitational force by the curvature of spacetime. Essentially, it explains the bending of a path in a gravitational field as being due to the curvature of spacetime, instead of to a force. The non-linear effects of this explanation fit observation better than the explanation by linear forces. However, our intuition continues to reside in the force concept, particularly when we have to include other forces in the discussion. For this reason one may want to reverse the procedure of GR and look at the non-linear "force of gravity" which would predict the same bending of the path as predicted by geometry. The problem is that the force depends upon the choice of frame. For any particular problem there may be one frame that is preferable to all others but, by the principle of general covariance, all frames are

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equally valid. As such, there is a need to identify the frame in which the non-linear "gravitational force" is defined and to specify in exactly what way it is preferable to others.

2. – The pseudo-Newtonian force

A very natural requirement, to be able to define a relativistic analogue of the Newtonian gravitational force, is that we have an inertial frame. This is provided by the freely falling frame in a Schwarzschild spacetime, for example. Of course, there is no gravitational force apparent in such a frame. To re-construct this force, one can observe the tidal force in the radial direction and work backwards to obtain the force of gravity that gives the observed tidal force. The problem with this procedure is that we have no observational way of knowing the "radial direction", in a general context, from within the inertial frame. As such, we need a more rigorously defined procedure.

The key to the procedure is the fact that the tidal force is maximum in the radial direction for the Schwarzschild spacetime. In geometric terms the tidal force is given by [1]

$$\mathscr{F}^{\mu} = -mR^{\mu}{}_{\nu\varrho\pi}\, t^{\nu} l^{\varrho} t^{\pi} \qquad (\mu, \nu \ldots = 0, \ldots, 3), \tag{1}$$

where m is the mass of the test particle, t^{ν} the timelike vector tangent to the particle's path and l^{ϱ} the separation vector, which provides the observation of the tidal force. In the free-fall frame $t^{\nu} = f\delta^{\nu}_{0}$, $f^{2} = 1/g_{00}$. Thus eq. (1) becomes

$$\mathscr{F}^{\mu} = -mf^{2} R^{\mu}{}_{0\varrho 0}\, l^{\varrho} \tag{2}$$

in this frame. Regarding this as an eigenvalue equation, we get the maximum tidal force along the eigenvector of the matrix $R^{\mu}{}_{0\varrho 0}$. Since l^{ϱ} is a purely spacelike vector in the free-fall frame, the maximum tidal force will be a purely spacelike vector. Thus we get

$$\overline{\mathscr{F}}^{i} = -mf^{2} R^{i}{}_{0j0}\, \overline{l}^{j}. \tag{3}$$

Here $\overline{l}^{j}$ provides the generalization of the "radial direction".

It can be shown [2] that this maximum tidal force can always be written as

$$\overline{\mathscr{F}}^{i} = \overline{l}^{j} F^{i}_{;j}, \tag{4}$$

where ";" stands for the covariant derivative. Thus F^{i} is the relativistic analogue of the Newtonian gravitational force. Further, this force is the gradient of a scalar quantity [3]

$$F_{i} = -\Phi_{,i}, \tag{5}$$

where an approximate expression for Φ is

$$\Phi = \frac{1}{2}(g_{00} - 1). \tag{6}$$

When expressed this way, a "Lorentz factor" has to be introduced by hand. A more appropriate expression is [4]

$$\Phi = \ln \sqrt{g_{00}}. \tag{7}$$

The F^i and Φ are called the *pseudo-Newtonian* (ψN) *force* and *potential*, respectively.

For the Kerr metric in Boyer-Lindquist coordinates [1], the ψN potential turns out to be [3]

$$\Phi = \ln \sqrt{(r^2 - 2Mr + a^2\cos^2\theta + Q^2)/(r^2 + a^2\cos^2\theta)}\,, \tag{8}$$

where M is the mass, Q the charge and a the angular momentum per unit mass, of the black hole (in units with $c = G = 1$). The corresponding ψN force is

$$\begin{cases} F_r = -(Mr^2 - Q^2 r - Ma^2\cos^2\theta)/(r^2 + a^2\cos^2\theta)^2, \\ F_\theta = a^2(2Mr - Q^2)\sin 2\theta / 2(r^2 + a^2\cos^2\theta)^2. \end{cases} \tag{9}$$

These quantities are defined in a special Fermi-normal frame corresponding to the rest frame of an observer falling freely from infinity in the radial direction. Due to the dragging of inertial frames [5], if the particle starts at an angle θ_0, it will spiral inward along a cone of half-angle θ_0.

In the above expressions, taking $Q = a = 0$, we get, for the Schwarzschild geometry, the usual Newtonian potential and force (with the appropriate Lorentz factor incorporation giving the potential as $\ln\sqrt{1 - 2M/r}$ instead of $-M/r$)! Taking $a = 0$, we get the relativistic modification due to the Reissner-Nordstrom geometry

$$\begin{cases} \Phi = \dfrac{1}{2}\ln(1 - 2M/r + Q^2/r^2) \approx -M/r(1 - Q^2/2Mr), \\ F_r = -M/r^2(1 - Q^2/Mr). \end{cases} \tag{10}$$

This leads to a *repulsion* of neutral matter by the charge of the black hole [6]. In the general case, there is a very rich structure to force [7], which is non-central. In fact, the "center" of the force appears to be a *ring in the equatorial plane* which one is tempted to identify with the ring singularity of the Kerr spacetime [8]! It leads to possible relativistic explanations for various phenomena [5, 9]. It also led to a better understanding of Carter's fourth invariant of motion [10] for a Kerr spacetime.

3. – The extended ψN force

One of the main reasons for developing the ψN formalism was that of being able to deal with the gravitational force like all other forces. Essentially, one has a choice of either trying to incorporate all forces into the geometry or removing even the gravitational force from it, if one wants to deal with all at a par [11]. In particular, any attempt at unification, with or without quantization, would seem to require such a goal. Thus we either go for Kaluza-Klein–like extensions of GR or a reformulation of GR in a ψN way, choosing a preferred frame on some physical grounds. For such a formulation to lead to a non-trivial field theory, the spacetimes dealt with should, in general, be non-static.

The major problem with the formulation of the ψN force and potential is that it is restricted to static spacetimes. An early attempt to avoid this restriction extended to conformally static spacetimes [12]. This extension gave consistent results but held out no hope of extending to genuinely non-static spacetimes required for a field theory which could resolve the unification and quantization problems.

In a genuinely time-varying spacetime, one expects to have a zero component of the force. However, for the eigenvalue problem there must be no zero component of the extremal tidal force. To obtain such an *extended* ψN(eψN) *force*, we are forced to give up the requirement of Fermi-normal coordinates for the time, while retaining them for the spatial, direction [13]. Subject to certain conditions, we obtain the eψN force

$$\left\{\begin{array}{l} F_0 = m[\{\ln(Af)\}_{,0} + g^{ik} g_{jk,0} g^{jl} g_{il,0}/4A], \\ F_j = m(\ln f)_{,i}, \end{array}\right. \tag{11}$$

where $A = [\ln\sqrt{-\det(g_{ij})}]_{,0}$.

This force formula depends on the choice of frame, which is not uniquely fixed. Thus, for the de Sitter spacetime, we get the usual cosmic repulsion, but when it is put into the Lemaitre form, we get $F_i = 0$ and $F_0 = m\sqrt{\Lambda/3}$. For the Friedmann models, again $F_i = 0$ and for the flat Friedmann, $F_0 = m/3t$. For the open and closed models, we get

$$\left\{\begin{array}{l} F_0^- = 2m/a_0 \sinh\eta(\cosh\eta - 1), \\ F_0^+ = m(4 + 3\sin^2\eta + 3\cos\eta + \cos^3\eta)/4a_0(1 - \cos\eta)\sin\eta, \end{array}\right. \tag{12}$$

where η is the usual Friedmann conformal time parameter [1]. This force is what would be experienced by a test particle in the given spacetime. To demonstrate the reality of gravitational waves, Ehlers and Kundt [14] considered a sphere of test particles in the path of a plane wave and demonstrated that momentum was imparted to the particles and the sphere was distorted. A similar analysis had been provided earlier by Weber and Wheeler [15] in the context of cylindrical waves. Using the above formula for the force that the particles experience and integrating over time gives the momentum, up to a constant of integration [13, 16]. Since the plane-fronted waves only impart a constant momentum and the eψN force given by eq. (11) is zero, we only get a consistency check from that case. However, for the cylindrical waves, we obtain a closed formula which, on expansion, yields the approximate results obtained by Weber and Wheeler! This provides a dramatic proof of the utility of the eψN force.

4. – Problems with the eψN formalism

In view of the success of the eψN formalism, one would hope that it could point us in the right direction for the unification and quantization attempts. As such, one needs as clear an understanding of its basis as one has of the ψN formalism. Unfortunately, there are some problems with it which need to be resolved.

The first problem is that the eψN force does not appear to be the 4-gradient of a single quantity to serve as the potential. Also, regarding the 4-force as the time derivative of a 4-momentum, we cannot identify the zero-component with the energy imparted to the test particle. *A clear understanding of this matter is required.*

Another problem is the identification of the frame in which the eψN force is observed. As pointed out, there is a non-uniqueness which led to two different answers for the force in the de Sitter geometry. There is a partial resolution of this problem [17]. The conditions, whose solution provides the eψN force, are

$$l^j(F^0_{,j} + \Gamma^0_{ij}F^i) = 0 \,, \tag{13}$$

$$l^j(F^i_{,j} + \Gamma^i_{0j}F^0) = mf^2 R^i_{0j0} l^j \,, \tag{14}$$

with the further requirements on the frame that $g_{ab,\,0i} = 0$ and $(\ln A)_{,\,0i} = 0$. These requirements exclude the Lemaitre form of the de Sitter metric. Hence the ambiguity in that case is avoided. In fact we are able, in some special cases, to determine the frame. However, there is no general proof that we *can* construct a unique frame in any spacetime, nor a proper understanding of what the frame is. *This understanding is of the utmost importance in making any progress with this approach to understand the implications of GR.*

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