On the singularity problem in cosmology (*)

G. MONTANI

ICRA, International Center for Relativistic Astrophysics
Dipartimento di Fisica, Università di Roma I "La Sapienza"
P.le A. Moro 2, I-00185 Roma, Italy

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Summary. — We present a critical analysis of the singularity problem in cosmology regarded in view of the evolution characterizing the universe near the initial "Big Bang". In particular, we enphasize how the asymptotic description of a generic inhomogeneous cosmological model singles out the appearance of a turbulent-like dynamics, requiring the introduction of an appropriate statistical method to be treated. To this end we write down the "Hopf equations" for the homogeneous Bianchi type-VIII and -IX models, thought as a test on the validity of the considered method.

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As pointed out in [1, 2] the time evolution of the Bianchi type-VIII and -IX homogeneous cosmological models, as well as of the general cosmological solution, is characterized asymptotically to the singularity by an iterative structure, whose single steps are based on the possibility of neglecting in the field equations the effects due to the spatial curvature.

Thus, in the leading order, such an asymptotic time evolution consists, for both the above cases, of an infinite sequence of "Kasner epochs", each of which is related to the previous one by well-defined transformation laws for all the quantities necessary to individualize a given Kasner-like regime.

Among these transformation laws of particular importance is the map relating the values of the Kasner indices in two consecutive steps of the iteration, whose form results to be the same for both the above-mentioned homogeneous and inhomogeneous cases, though with the non-trivial difference that in the latter one such map should be

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applicated point by point in space due to the dependence of the Kasner indices on the spatial coordinates.

This famous result, obtained by Belinskii, Lifshitz and Kalatnikov (BKL), constitutes probably the most important achievement of relativistic cosmology, but while there cannot exist any controversy about its validity with reference to the Bianchi type-VIII and -IX models, its extension to the behaviour of a generic inhomogeneous cosmological model leaves some important questions open. In order to discuss in some detail the nature of such questions, we give a very brief review of the BKL approach to this general inhomogeneous case.

In a synchronous reference (coordinates t and x^γ ($\gamma = 1, 2, 3$)) each single step of the above-mentioned iterative structure, singled out by the system evolution to the singularity ($t = 0$), corresponds to a three-dimensional metric tensor which, in the leading order, takes the form

$$\gamma_{\alpha\beta}(t, x^\gamma) = t^{2p_l(x^\gamma)} l_\alpha l_\beta + t^{2p_m(x^\gamma)} m_\alpha m_\beta + t^{2p_n(x^\gamma)} n_\alpha n_\beta ,$$

where, **l**, **m** and **n** denote three linear independent vectors whose components are arbitrary functions of the spatial coordinates while the Kasner indices $p_a(x^\gamma)$ ($a = l, m, n$) verify the two following conditions:

$$p_l(x^\gamma) + p_m(x^\gamma) + p_n(x^\gamma) = p_l(x^\gamma)^2 + p_m(x^\gamma)^2 + p_n(x^\gamma)^2 = 1 .$$

Point by point in space such functions can be parametrized in terms of a single one, $u(x^\gamma)(1 \leqslant u < \infty)$, whose time evolution admits a piecewise representation based on the well-known BKL transformation law [1-3]

$$u' = u - 1 \quad \text{if } u \geqslant 2 , \qquad u' = 1/(u-1) \quad \text{if } 1 \leqslant u < 2 . \tag{1}$$

In [3, 4] there is shown how combining the sensitivity of the BKL map on the initial data, with its pointwise character, leads in a natural way to the appearance of an oscillating spatial dependence for the Kasner index functions $p_a(x^\gamma)$. Such a phenomenon (called in [3] *fragmentation process*), due to the coupling existing in the system, involves the whole spatial dependence of the three-dimensional metric tensor and therefore, when considering the general solution, any assigned degree of inhomogeneity, even arbitrarily small, is expected to evolve asymptotically into a completely “disordered” configuration of the spatial slices.

In particular, it is possible to show [3] that, as an effect of the fragmentation process, asymptotically to the cosmological singularity the three-dimensional metric tensor tends to assume, point by point in space, an expression of the form

$$\gamma_{\alpha\beta}(t, u) = \sqrt[3]{\gamma}\, e^{H_{\alpha\beta}} = f(t, u)\, g(t)\, e^{H_{\alpha\beta}(t, u)} , \tag{2}$$

where $f(t, u)$ denotes a wave-like function, $g(t)$ a monotonically decreasing one and $H_{\alpha\beta}(t, u)$ a traceless symmetric matrix.

With reference to a fixed point of space the map (1) behaves as the corresponding one in the homogeneous case and therefore, asymptotically, the integer part K and the fractional one X of the parameter u admit the joint probability distribution discussed in [5, 6]. Since, by definition, $u = K + X$, this result can be restated in terms of a stationary probability distribution for the parameter u which takes the form

$$w(u) = \frac{1}{u(u+1)\ln 2}\,. \tag{3}$$

Combining together the results (2) and (3) we see that, at each fixed point of space, the three-dimensional metric tensor acquires an asymptotic random behaviour.

A more detailed statistical treatment for the chaotic evolution performed by the spatial slices, when regarded as a whole, can be found in [7].

As pointed out in [3, 8], due to the nonlinearity of the field equations, the ever increasing spatial gradients (diverging for $t \to 0$) continuously created by the fragmentation process can induce a dynamical instability of the oscillatory regime with a consequent broadening of the spectrum characterizing the asymptotic gravitational field. Furthermore, since the appearance of self-organization phenomena should be regarded as improbable, there is expected a transition of the system to a new dynamical regime whose feature is yet, reliably, a turbulent-type one.

Indeed the question about which is the detailed structure of this turbulent behaviour, performed by a generic gravitational field in such a considered extreme regime, is yet substantially open, subjected to the possibility of taking precise account for the asymptotic role played by the spatial gradients.

Since each solution of the Einstein equations can be formally represented as a trajectory in the *phase superspace* (*i.e.* the space of the initial data for the gravitational Chauchy problem), we can regard the appearance of this chaotic evolution like the outcoming indetermination due to the "ergodicity" shown by the system in such a space.

By analogous way, the chaotic time evolution of the Bianchi type-VIII and -IX models finds its natural interpretation in terms of the complexity of their *phase minisuperspace* trajectories.

Indeed, on the base of our actual knowledge of the considered systems, we cannot say which precise kind of "ergodic property" takes place in the *phase superspace*, and in the absence of a proved "ergodic theorem", the possibility of replacing the time average along the phase trajectory by an average over a suitable ensamble, should be regarded simply as an assumption.

Once such an assumption is made, we can introduce, in standard way [9], space or space-time probability distributions of any order, by which calculating, in principle, all the average quantities of interest for the system dynamics.

Here we discuss the application to the gravitational case of the same functional method which in fluids dynamics leads to write down the so-called Hopf equations for the velocity field, and whose structure can be summarized as follows.

Assigned a given space-time–dependent vector random field ([1]) $\vec{X}(t, x^{\gamma})$, the above method is based on the introduction of the so-called characteristic functional of the system, obtained as the infinite-dimensional limit of the Fourier transformal of the finite-dimensional probability densities, *i.e.* the functional Fourier transformal of the probability functional. An object such constructed can take two different versions, space or space-time [9], depending on whether the random field is regarded as a purely spatial one, having fixed a precise value of the time variable, or in view of its global

([1]) The vectorial nature of the field is, of course, not essential to the applicability of the method and has been chosen to make simpler the comparison with [9].

space-time dependence. In any case the characteristic functional can be schematically represented as follows:

$$\Phi(\vec{Y}) = \left\langle \exp\left[i \int_{-\infty}^{+\infty} (\vec{Y}\cdot\vec{X}) \right] \right\rangle, \tag{4}$$

where $\vec{Y}$ is the conjugated field to the random one and, according to what said above, it can be space or space-time dependent. Here the notation $\langle\ \rangle$ denotes the mean value of the enclosed quantity, $(\cdot)$ the scalar product of the two considered vector variables and $\int_{-\infty}^{+\infty}$ an integration, respectively, over the whole space or space-time domain available to the system.

For a characteristic functional so defined it is possible to write down a system of functional differential equations which translates into the new language the dynamical content of the Einstein ones. Once solved such equations we can, in principle, calculate the probability functional of the system by taking the Fourier transformal of the characteristic one, *i.e.*

$$\Psi(\vec{X}) = \int \left\{ \exp\left[i \int_{-\infty}^{+\infty} (\vec{X}\cdot\vec{Y}) \right] \Phi(\vec{Y})\, \mathrm{d}\mu(\vec{Y}) \right\},$$

where $\mu(\vec{Y})$ denotes a measure in the infinite-dimensional function space.

In what follows we apply this method to the description of the chaotic time evolution performed by the Bianchi type-VIII and -IX models near the cosmological singularity. The possibility of reproducing the BKL results for such a simple homogeneous case should be regarded as a test for the validity of the method even in the general one.

The dynamical evolution of such models is summarized [1] by the following ordinary first-order differential system with respect the time variable:

$$\begin{cases} \alpha_{,\tau} = p_\alpha; \qquad \beta_{,\tau} = p_\beta; \qquad \gamma_{,\tau} = p_\gamma, \\ p_{\alpha,\tau} = \dfrac{1}{2}\left[(e^{2\beta} - e^{2\gamma})^2 - e^{4\alpha} \right], \\ p_{\beta,\tau} = \dfrac{1}{2}\left[(e^{2\alpha} \mp e^{2\gamma})^2 - e^{4\beta} \right], \\ p_{\gamma,\tau} = \dfrac{1}{2}\left[(e^{2\alpha} \mp e^{2\beta})^2 - e^{4\gamma} \right], \end{cases} \tag{5}$$

where $(\ )_{,\tau} \equiv \mathrm{d}(\ )/\mathrm{d}\tau$ and $(+)$ refers to the type VIII, while $(-)$ to the type IX.

Furthermore, this system admits a first integral of the form

$$\frac{1}{2}(p_\alpha + p_\beta + p_\gamma)_{,\tau} = p_\alpha p_\beta + p_\alpha p_\gamma + p_\beta p_\gamma . \tag{6}$$

If, from a purely formal point of view, we interpret the three logaritmic scale factors and their time derivatives as the components of two vectors $\vec{u}(\tau)$, $\vec{p}(\tau)$, *i.e.*

$$(\alpha, \beta, \gamma) \equiv (u_1, u_2, u_3), \qquad (p_\alpha, p_\beta, p_\gamma) \equiv (p_1, p_2, p_3),$$

then the definition of an appropriate characteristic functional of the space-time type (though in this case the spatial variables are not involved in the problem) for such system, results to be, in agreement with (4), as follows:

$$\Phi(\theta_1, \theta_2, \theta_3, \eta_1, \eta_2, \eta_3) \equiv \langle e^{i[(\vec{\theta}\cdot\vec{u}) + (\vec{\eta}\cdot\vec{p})]}\rangle, \tag{7}$$

where $\vec{\theta}(\tau) \equiv (\theta_1, \theta_2, \theta_3)$ and $\vec{\eta}(\tau) \equiv (\eta_1, \eta_2, \eta_3)$ denote the conjugated variables, $\langle\ \rangle$, as before, the mean value of the enclosed quantity and the notation

$$(\vec{\theta}\cdot\vec{u}) \equiv \int_{-\infty}^{+\infty}\left(\sum_{i=1}^{3}\theta_i u_i\right)\mathrm{d}\tau, \qquad (\vec{\eta}\cdot\vec{p}) \equiv \int_{-\infty}^{+\infty}\left(\sum_{i=1}^{3}\eta_i p_i\right)\mathrm{d}\tau$$

has been assumed.

Denoting by $\mathscr{D}_{\theta_i}$ and $\mathscr{D}_{\eta_i}$ $(i = 1, 2, 3)$ the ordinary functional derivatives with respect to the conjugated variables, it is easy to realize that the following key relations take place:

$$\mathscr{D}_{\theta_i}\Phi = \langle i u_i e^{i[(\vec{\theta}\cdot\vec{u}) + (\vec{\eta}\cdot\vec{p})]}\rangle,$$

$$\mathscr{D}_{\eta_i}\Phi = \langle i p_i e^{i[(\vec{\theta}\cdot\vec{u}) + (\vec{\eta}\cdot\vec{p})]}\rangle,$$

from which, observing that the exponential term is time independent, we get

$$(\mathscr{D}_{\theta_i}\Phi)_{,\tau} = \langle i u_{i,\tau} e^{i[(\vec{\theta}\cdot\vec{u}) + (\vec{\eta}\cdot\vec{p})]}\rangle,$$

$$(\mathscr{D}_{\eta_i}\Phi)_{,\tau} = \langle i p_{i,\tau} e^{i[(\vec{\theta}\cdot\vec{u}) + (\vec{\eta}\cdot\vec{p})]}\rangle.$$

By using the field equations (5) and taking into account that

$$\langle e^{u_i} e^{i[(\vec{\theta}\cdot\vec{u}) + (\vec{\eta}\cdot\vec{p})]}\rangle = e^{-i\mathscr{D}_{\theta_i}}\Phi \qquad (i = 1, 2, 3)$$

we finally see that the space-time version of the "Hopf equations" for the homogeneous Bianchi type-VIII and -IX cosmological models reads in the form

$$\mathscr{D}_{\eta_i}\Phi = (\mathscr{D}_{\theta_i}\Phi)_{,\tau} \qquad (\mathscr{D}_{\eta_i}\Phi)_{,\tau} = \Lambda_i\Phi \qquad (i = 1, 2, 3), \tag{8}$$

where by Λ_i $(i = 1, 2, 3)$ we denoted the following functional differential operators:

$$\Lambda_1 \equiv \frac{i}{2}[(e^{-2i\mathscr{D}_{\theta_2}} - e^{-2i\mathscr{D}_{\theta_3}})^2 - e^{-4i\mathscr{D}_{\theta_1}}],$$

$$\Lambda_2 \equiv \frac{i}{2}[(e^{-2i\mathscr{D}_{\theta_1}} \mp e^{-2i\mathscr{D}_{\theta_3}})^2 - e^{-4i\mathscr{D}_{\theta_2}}],$$

$$\Lambda_3 \equiv \frac{i}{2}[(e^{-2i\mathscr{D}_{\theta_1}} \mp e^{-2i\mathscr{D}_{\theta_2}})^2 - e^{-4i\mathscr{D}_{\theta_3}}].$$

Now observing that such a vector operator $\vec{\Lambda}$ consists only of translational operators in the θ-space, the above equations can be formally rewritten in the following, more extensive, form:

$$
(9) \quad \begin{cases}
\mathscr{D}_{\eta_1}\Phi = (\mathscr{D}_{\theta_1}\Phi)_{,\tau}, \qquad \mathscr{D}_{\eta_2}\Phi = (\mathscr{D}_{\theta_2}\Phi)_{,\tau}, \qquad \mathscr{D}_{\eta_3}\Phi = (\mathscr{D}_{\theta_3}\Phi)_{,\tau} \\
(\mathscr{D}_{\eta_1}\Phi)_{,\tau} = \dfrac{i}{2}[\Phi(\theta_1, \theta_2 - 4i, \theta_3, \vec{\eta}) + \Phi(\theta_1, \theta_2, \theta_3 - 4i, \vec{\eta}) - \\
\qquad - \Phi(\theta_1 - 4i, \theta_2, \theta_3, \vec{\eta}) - 2\Phi(\theta_1, \theta_2 - 2i, \theta_3 - 2i, \vec{\eta})], \\
(\mathscr{D}_{\eta_2}\Phi)_{,\tau} = \dfrac{i}{2}[\Phi(\theta_1 - 4i, \theta_2, \theta_3, \vec{\eta}) + \Phi(\theta_1, \theta_2, \theta_3 - 4i, \vec{\eta}) - \\
\qquad - \Phi(\theta_1, \theta_2 - 4i, \theta_3, \vec{\eta}) \mp 2\Phi(\theta_1 - 2i, \theta_2, \theta_3 - 2i, \vec{\eta})], \\
(\mathscr{D}_{\eta_3}\Phi)_{,\tau} = \dfrac{i}{2}[\Phi(\theta_1 - 4i, \theta_2, \theta_3, \vec{\eta}) + \Phi(\theta_1\theta_2 - 4i, \theta_3, \vec{\eta}) - \\
\qquad - \Phi(\theta_1, \theta_2, \theta_3 - 4i, \vec{\eta}) \mp 2\Phi(\theta_1 - 2i, \theta_2 - 2i, \theta_3, \vec{\eta})].
\end{cases}
$$

It is easy to see that using the functional (7) the first integral (6) will be rewritten as

$$
(10) \qquad [(\mathscr{D}_{\eta_1} + \mathscr{D}_{\eta_2} + \mathscr{D}_{\eta_3})\,\Phi]_{,\tau} = -2i(\mathscr{D}_{\eta_1}\mathscr{D}_{\eta_2} + \mathscr{D}_{\eta_1}\mathscr{D}_{\eta_3} + \mathscr{D}_{\eta_2}\mathscr{D}_{\eta_3})\,\Phi\,.
$$

Clearly this functional equation plays the role of a differential constraint for the systems (8) or (9).

The Fourier transformal of (7) gives the probability functional $\Psi(u_1, u_2, u_3, p_1, p_2, p_3)$, which associates a well-defined probability amplitude to each trajectory in the *phase minisuperspace*.

Though such a space-time version of the equations gives the most complete representation of the statistical properties of the system, nevertheless it provides more information than it is really necessary since it involves the whole “dynamical history” of the system, combining together integrable and non-integrable evolutive stages. This fact makes the space-time approach to the problem rather unsuitable to the reproduction of the BKL results.

Let us now face the study of the space version of the “Hopf equations” for the very same cosmological models. In such a case, since the problem involves only the dependence on the time variable, the characteristic functional does not contain any integration on the spatial variables, so reducing to a simple characteristic function of the form

$$
(11) \qquad \phi(\theta_1, \theta_2, \theta_3, \eta_1, \eta_2, \eta_3, \tau) = \langle e^{i\sum_{i=1}^{3}[\theta_i u_i(\tau) + \eta_i p_i(\tau)]}\rangle,
$$

where $\vec{\theta}$ and $\vec{\eta}$ now denote two constant vector fields.

By taking the first time derivative of such a function we easily obtain

$$
\phi_{,\tau} = \left\langle \sum_{i=1}^{3} i[\theta_i u_{i,\tau} + \eta_i p_{i,\tau}]\, e^{i\sum_{i=1}^{3}[\theta_i u_i(\tau) + \eta_i p_i(\tau)]} \right\rangle.
$$

Now, using again the dynamical equations for the system and rewriting the expressions so obtained in terms of the characteristic function (11), we arrive at the following basic equation:

$$\phi_{,\tau} = \sum_{i=1}^{3} (\theta_i \partial_{\eta_i} \phi + \eta_i \lambda_i \phi), \tag{12}$$

where $\vec{\lambda}$ denotes the ordinary partial differential vector operator obtained from $\vec{\Lambda}$ by replacing the functional derivatives $\mathscr{D}_{\theta_i}$ with the ordinary partial derivatives ∂_{θ_i} ($i = 1, 2, 3$).

Since even $\vec{\eta}$ consists only of translational operators in the parametric θ-space, we reformulate the above equation in the more extensive form:

$$\begin{aligned}\phi_{,\tau} = {} & (\theta_1 \partial_{\eta_1} + \theta_2 \partial_{\eta_2} + \theta_3 \partial_{\eta_3}) \phi + \frac{i}{2} [(-\eta_1 + \eta_2 + \eta_3) \phi(\theta_1 - 4i, \theta_2, \theta_3, \vec{\eta}, \tau) + \\ & + (\eta_1 - \eta_2 + \eta_3) \phi(\theta_1, \theta_2 - 4i, \theta_3, \vec{\eta}, \tau) + (\eta_1 + \eta_2 - \eta_3) \phi(\theta_1, \theta_2, \theta_3 - 4i, \vec{\eta}, \tau) - \\ & - 2\eta_1 \phi(\theta_1, \theta_2 - 2i, \theta_3 - 2i, \vec{\eta}, \tau) \mp 2\eta_2 \phi(\theta_1 - 2i, \theta_2, \theta_3 - 2i, \vec{\eta}, \tau) \mp \\ & \mp \eta_3 \phi(\theta_1 - 2i, \theta_2 - 2i, \theta_3, \vec{\eta}, \tau)].\end{aligned} \tag{13}$$

Finally, in terms of the characteristic function (11) the first integral (6) reads in the form

$$(\partial_{\eta_1} \partial_{\eta_2} + \partial_{\eta_1} \partial_{\eta_3} + \partial_{\eta_2} \partial_{\eta_3}) \phi = \frac{i}{2} (\lambda_1 + \lambda_2 + \lambda_3) \phi . \tag{14}$$

Of course, even in this case such an equation should be regarded simply as a differential constraint for the systems (12) or (13).

The Fourier transformal of the characteristic function (11) gives the evolution in time of the probability amplitude associated to each single point of the *phase minisuperspace*.

As is easy to realize, due to its evolutive character, this space version of the "Hopf equations" represents the natural tool for a statistical description of the Bianchi type-VIII and -IX models in view of a direct comparison with the BKL results.

In this respect we observe that the solution of the above systems (12) and (13) (together with the constraint (14)) should be subjected to a Chauchy problem of the form

$$\phi(\vec{\theta}, \vec{\eta}, \tau_0) = \phi_0(\vec{\theta}, \vec{\eta}),$$

where ϕ_0 should be subjected only to the constraint (14).

We conclude our reformulation of the considered problem by stressing how the translation of the field equations in the new statistical language leads to the derivation of a linear theory for the system dynamics. It is just in the appearance of such a linear dynamics that this statistical method finds the reasons of its implementation.

Now we discuss some important features characterizing the homogeneous mixmaster model as it results by the BKL approach. In the first place, we should observe how such an approach describes simply the statistical behaviour of variables whose nature is characterized by intrinsic properties rather than by belonging to a fixed direction. Thus, for instance, in [5, 6] the evolution of the parameter u (*i.e.* of the ordered Kasner indices) and of the oscillations amplitude is discussed.

In particular, it is shown, in agreement with (3), that the ordered Kasner indices admit a stationary statistical distribution. Since in each Kasner epoch these indices coincide, in the leading order, with the field

$$p_i^*(\tau) = \text{ordering of } \{p_i(\tau)\},$$

we can conclude that it is a stationary random one, as well as that its integral field $u_i^*(\tau)$ is characterized by stationary increments and therefore its mean value should take a linear behaviour in time, *i.e.*

$$\langle u_i^* \rangle = v_i \tau, \qquad v_i = \text{const}.$$

But from the analysis developed in [5, 6] it comes out in a very clear way how the quantities v_i result to be divergent and, consequently, the field u_i^* does not possess a stable statistical distribution.

Since such features hold also for the field $u_i(\tau)$, the implementation of the above statistical method in terms of these fields, although it is the most natural, is nevertheless appropriate only to the description of the derivative field.

A suitable set of variables which admit, as a whole, a stationary statistical distribution can be introduced by reformulating the dynamical problem in terms of the well-known Misner-Chitré variables [10]. Such a reformulation is to be developed and a detailed discussion of the "Hopf equations" relative to the above models, as described in terms of these appropriate variables, will be presented in the future elsewhere.

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