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Canonical quantization of (2 + 1)-dimensional gravity coupled to spinning particles (*)

T. LEE (1) and S. ZOH (2)

(1) *Department of Physics, Kangwon National University - Chuncheon 200-701, Korea*
(2) *Center for Theoretical Physics, Seoul National University - Seoul 151-742, Korea*

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Summary. — We construct an action and carry out its canonical quantization for (2 + 1)-dimensional gravity with the gravitational Chern-Simons term, coupled to point particles with spins. While doing so we use the coadjoint-orbit method and apply the Faddeev-Jackiw first-order formalism. In addition, we obtain the braiding relation and discuss the scattering process of the two-particle case.

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Quantum gravity is one of the most important and challenging problems in contemporary theoretical physics. Although there have been a lot of attempts, we do not have a definite answer to it, yet. In this respect, three-dimensional gravity provides a good toy model, where the mathematical structures are greatly simplified but still rich enough for us to gain much insight into the interrelation between geometry and quantum mechanics, which may be an indispensable ingredient toward a consistent and complete theory of quantum gravity. This virtue of the three-dimensional gravity can be mostly ascribed to the fact that the three-dimensional gravity can be described as a non-Abelian Chern-Simons gauge theory of which the gauge group is the three-dimensional Poincaré group $ISO(2, 1)$ [1, 2].

In this paper we will investigate, in the framework of Chern-Simons gauge-theoretic formulation, a *generalized* three-dimensional gravity coupled to point particles with arbitrary spins, which incorporates the *gravitational Chern-Simons term*. The gravitational Chern-Simons term originates from the unique feature of three-dimensional space-time: its Poincaré group $ISO(2, 1)$ has one-parameter family

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of invariant non-degenerate inner products [3]. On the other hand, we will exploit the coadjoint-orbit method [4, 5] to construct the action of the point particles which keeps the local Poincaré symmetry manifest, and will take advantage of Faddeev-Jackiw's first-order formalism [6] to perform canonical quantization. And we will compare it with the theories of three-dimensional gravity with the conventional inner product [7, 8]. We start by constructing the generalized invariant non-degenerate inner product on $ISO(2,1)$. Let us first consider the algebra defined by

$$[P_a, P_b] = -\lambda \varepsilon_{ab}{}^c J_c, \qquad [P_a, J_b] = \varepsilon_{ab}{}^c P_c, \qquad [J_a, J_b] = \varepsilon_{ab}{}^c J_c, \tag{1}$$

where $a, b, c = 0, 1, 2$. The above algebra corresponds to $SO(3,1)$, $SO(2,2)$, or $ISO(2,1)$ if $\lambda > 0$, $\lambda < 0$, or $\lambda = 0$, respectively. One of the most interesting features in three-dimensional space-time, which is indeed the underlying basis allowing a gauge-theoretic formulation for three-dimensional gravity, is that the above algebra admits *two* invariant inner products as follows:

$$\begin{cases} \langle P_a, P_b \rangle = 0\,, \\ \langle P_a, J_b \rangle = \eta_{ab}, \\ \langle J_a, J_b \rangle = 0\,, \end{cases} \tag{2}$$

or

$$\begin{cases} \langle P_a, P_b \rangle = \lambda \eta_{ab}, \\ \langle P_a, J_b \rangle = 0\,, \\ \langle J_a, J_b \rangle = \eta_{ab}, \end{cases} \tag{3}$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, 1)$. Notice that the inner product (2), which is non-degenerate, is peculiar only to three-dimensional space-time, although the inner product (3) has its natural counterparts in any dimensional space-time, which are unfortunately degenerate if $\lambda = 0$. We would like to emphasize once more that the above fact is the very reason why only three-dimensional gravity exhibits a simple gauge-theoretic formulation while others do not. Now we can construct the most *general* invariant non-degenerate inner product on the algebra by combining the above two inner products as $(2) + \alpha(3)$, where α is an arbitrary real number. Putting $\lambda = 0$, we have at last arrived at the generalized invariant non-degenerate inner product on $ISO(2,1)$, defined as

$$\begin{cases} \langle P_a, P_b \rangle = 0\,, \\ \langle P_a, J_b \rangle = \eta_{ab}, \\ \langle J_a, J_b \rangle = \alpha \eta_{ab}. \end{cases} \tag{4}$$

In the limit α goes to zero, the above inner product obviously reduces to the conventional one previously used in the Chern-Simons gauge formulation of three-dimensional gravity [2].

Using the above generalized inner product for the Lie-algebra–valued gauge potential of $ISO(2,1)$ defined as

$$A = A_\mu \mathrm{d}x^\mu, \qquad A_\mu = e_\mu{}^a P_a + \omega_\mu{}^a J_a, \tag{5}$$

we obtain the *generalized* Chern-Simons action for gravity in three dimensions as [9]

$$\text{(6)} \qquad \begin{cases} I_{\mathrm{G}} = \int \mathrm{d}t\, L_{\mathrm{G}}, \\ L_{\mathrm{G}} = \dfrac{1}{2}\int \mathrm{d}^2 x \left\langle A,\, \mathrm{d}A + \dfrac{2}{3}A^2 \right\rangle = \int \mathrm{d}^2 x \varepsilon^{\lambda\mu\nu} e_\lambda{}^c (\partial_\mu \omega_\nu{}^c - \partial_\nu \omega_\mu{}^c + \varepsilon_{abc}\omega_\mu{}^a \omega_\nu{}^b) + \\ \qquad + \dfrac{\alpha}{2}\int \mathrm{d}^2 x \varepsilon^{\lambda\mu\nu} \omega_\lambda{}^c \left(\partial_\mu \omega_\nu{}^c - \partial_\nu \omega_\mu^c + \dfrac{2}{3}\varepsilon_{abc}\omega_\mu{}^a \omega_\nu{}^b\right). \end{cases}$$

We note that the obtained Lagrangian contains the gravitational Chern-Simons term in addition, being compared with the previous one. However, the above Lagrangian yields the same equations of motion as the Einstein-Cartan theory of three-dimensional gravity when the matter is absent.

For the particles, we take the following action which manifests the local gauge symmetry $ISO(2,1)$:

$$\text{(7)} \qquad \begin{cases} L_{\mathrm{P}} = \left\langle K,\, g^{-1}\left(\dfrac{\partial}{\partial t} + A_{\mathrm{t}}\right) g\right\rangle, \\ K = mJ^0 + sP^0, \\ A_{\mathrm{t}} = e_{\mathrm{t}}{}^a P_a + \omega_{\mathrm{t}}{}^a J_a = \dot{\xi}^\mu e_\mu{}^a P_a + \dot{\xi}^\mu \omega_\mu{}^a J_a\,, \end{cases}$$

where g is an element of $ISO(2,1)$ and ξ^μ denote the space-time coordinates of the particle. With expressing g in terms of the Euler angles ϕ, θ, ψ and a Lorentz vector q^a, $g(t) = (\Lambda, q)$, $\Lambda = e^{J_0\phi} e^{J_1\theta} e^{J_0\psi}$ and performing some modifications, learnt from the lesson in the relativistic anyon theory [10], which helps greatly for the canonical quantization, we find the Lagrangian for N point particles with arbitrary spins coupled to gravity as [9]

$$\text{(8)} \qquad L_{\mathrm{P}} = \sum_A [\pi_{A\mu}\dot{\xi}^\mu_A + p_{Aa}\dot{q}^a_A + \bar{s}_A \cosh\theta_A \dot{\phi}_A + \lambda_A (p_{Aa}\bar{S}^a_A + \bar{s}_A m_A) + N_A (p_A^2 + m_A^2)] + \\ + \sum_A \beta^\mu [\pi_{A\mu} - \bar{S}_{Aa}\omega_\mu{}^a - p_{Aa}(e_\mu{}^a + \varepsilon^a{}_{bc}\omega_\mu{}^b q^c)]\,,$$

where $A = 1, \ldots, N$ labels the particles. It will be clear shortly that the parameters m_A and $\bar{s}_A = s_A + \alpha m_A$ correspond to mass and spin of the particles, respectively.

If we set $\xi^0_A = t$, using the diffeomorphism invariance, we have an expression of the Lagrangian of the gravity coupled to N point particles with spins, which is more suitable for the canonical quantization

$$\text{(9)} \qquad L = L_{\mathrm{G}} + L_{\mathrm{P}} = \sum_A [\pi_{Ai}\dot{\xi}^i_A + p_{Aa}\dot{q}^a_A + \bar{s}_A \cosh\theta_A \dot{\phi}_A + N_A \chi^A_{\mathrm{M}} + \lambda_A \chi^A_{\mathrm{PL}} + \beta^i_A \chi^A_i] + \\ + \int \mathrm{d}^2 x [\varepsilon^{ij}\omega_j{}^a (2\,\dot{e}_{ia} + \alpha\dot{\omega}_{ia}) + e_{0a}\varphi^a_{\mathrm{R}} + \omega_{0a}\varphi^a_{\mathrm{T}}]\,.$$

As usual for a reparametrization-invariant theory, the Hamiltonian consists of constraints only:

$$
(10)\qquad \begin{cases}
\varphi^a_{\mathrm{R}}(\mathbf{x}) = \varepsilon^{ij} R_{ij}{}^a(\mathbf{x}) + \sum_A p^a_A\, \delta(\mathbf{x} - \boldsymbol{\xi}_A) = 0\,, \\
\varphi^a_{\mathrm{T}}(\mathbf{x}) = \varepsilon^{ij} (T^a_{ij}(\mathbf{x}) + \alpha R^a_{ij}(\mathbf{x})) + \sum_A J^a_A\, \delta(\mathbf{x} - \boldsymbol{\xi}_A) = 0\,, \\
\chi^A_i = \pi_{Ai} - J_{Aa}\, \omega^a_i(\boldsymbol{\xi}_A) - p_{Aa}\, e_i{}^a(\boldsymbol{\xi}_A) = 0\,, \\
\chi^A_{\mathrm{M}} = p^2_A + m^2_A = 0\,, \\
\chi^A_{\mathrm{PL}} = p_{Aa} \overline{S}^a_A + \bar{s}_A\, m_A = 0\,,
\end{cases}
$$

where $J^a_A = L^a_A + \overline{S}^a_A = \varepsilon^a_{bc} q^b_A p^c_A + \overline{S}^a_A$, and

$$
R_{ij}{}^c = \partial_i \omega_j{}^c - \partial_j \omega_i{}^c + \varepsilon_{ab}{}^c \omega_i{}^a \omega_j{}^b\,,
$$

$$
T_{ij}{}^c = \partial_i e^c_j - \partial_j e^c_i + \varepsilon_{ab}{}^c (\omega_i{}^a e^b_j - \omega_j{}^a e^b_i)\,,
$$

which are spatial components of the curvature and the torsion, respectively. Notice that the constraints χ^A_{M} and χ^A_{PL} are nothing but the mass-shell condition and the Pauli-Lubanski condition, respectively, which are just what we need to describe the relativistic anyons. Since the mass and the spin are determined by the mass-shell condition and the Pauli-Lubanski condition, we find that the spin of the anyon is shifted by αm due to the gravitational Chern-Simons term. It is not difficult to see that when $s_A = \alpha = 0$ the above Lagrangian (9) reduces to that of the approach [7, 8] which takes advantage of the Poincaré coordinates; q^a_A are nothing but the Poincaré coordinates.

Applying Faddeev-Jackiw's first-order formalism [6], we can easily show that the above Lagrangian yields the fundamental commutators as

$$
(11)\qquad \begin{cases}
\{\xi^i_A, \pi_{Bj}\} = \delta^i{}_j\, \delta_{AB}\,, \qquad \{q^a_A, p^b_B\} = \eta^{ab}\, \delta_{AB}\,, \\
\{\overline{S}_{Aa}, \overline{S}_{Bb}\} = \varepsilon_{ab}{}^c \overline{S}_{Ac}\, \delta_{AB}\,, \qquad \{e_i{}^a(\mathbf{x}), \omega_j{}^b(\mathbf{y})\} = \dfrac{1}{2}\, \varepsilon_{ij}\, \eta^{ab}\, \delta^2(\mathbf{x} - \mathbf{y})\,, \\
\{e_i{}^a(\mathbf{x}), e_j{}^b(\mathbf{y})\} = -\dfrac{\alpha}{2}\, \varepsilon_{ij}\, \eta^{ab}\, \delta^2(\mathbf{x} - \mathbf{y})\,, \qquad \{\omega_i{}^a(\mathbf{x}), \omega_j{}^b(\mathbf{y})\} = 0\,.
\end{cases}
$$

From these fundamental commutators, we find that the constraints form a closed algebra. The only non-vanishing commutators of them are as follows:

$$
(12)\qquad \begin{cases}
\{\varphi^a_{\mathrm{R}}(\mathbf{x}), \varphi^b_{\mathrm{R}}(\mathbf{y})\} = 0\,, \\
\{\varphi^a_{\mathrm{T}}(\mathbf{x}), \varphi^b_{\mathrm{R}}(\mathbf{y})\} = \varepsilon^{ab}{}_c\, \varphi^c_{\mathrm{R}}(\mathbf{x})\, \delta^2(\mathbf{x} - \mathbf{y})\,, \\
\{\varphi^a_{\mathrm{T}}(\mathbf{x}), \varphi^b_{\mathrm{T}}(\mathbf{y})\} = \varepsilon^{ab}{}_c\, \varphi^c_{\mathrm{T}}(\mathbf{x})\, \delta^2(\mathbf{x} - \mathbf{y})\,, \\
\{\chi^A_i, \chi^B_j\} = \dfrac{1}{2}\, \delta^{AB}\, \varepsilon_{ij} [(J_a - \alpha p_a)\, \varphi^a_{\mathrm{R}} + p_a\, \varphi^a_{\mathrm{T}}]\,.
\end{cases}
$$

As expected, the constraints φ^a_{R} and φ^a_{T} form an $ISO(2, 1)$ algebra. The constraints χ^A_i are present because of the underlying diffeomorphism invariance of the theory, which cannot be completely represented by the action of the $ISO(2, 1)$ constraints when it acts by moving particles around [8].

After having constructed a classical action for the three-dimensional gravity with the gravitational Chern-Simons term, coupled to point particles with spins, and performed its canonical quantization, we would like to end the paper by deriving the braiding operator and scattering amplitude. Although it is rather straightforward to apply most of the apparatus already developed for the NACS particles [11-13] to the spinning particles coupled with the gravity under discussion, in this paper we shall consider only the simplest case of a two-particle sector, especially in the Coulomb gauge, which cannot possibly be chosen among the sectors with an arbitrary number of particles. Let us suppose that one particle (labeled by 1) is located at the origin and the other (labeled by 2) is at $\mathbf{x}$. Choosing a Coulomb gauge condition, $\partial_i A_i{}^a = 0$, or equivalently,

$$\partial_i e_i^a = \partial_i \omega_i{}^a = 0\,,$$

we easily find a solution of the Gauss constraint,

$$e_i{}^a(\mathbf{x}) = \varepsilon_{ij} \frac{\widehat{J}_1^a + \alpha \widehat{p}_1^a}{2\pi} \frac{x^j}{r^2}\,, \qquad \omega_i{}^a(\mathbf{x}) = \varepsilon_{ij} \frac{\widehat{p}_1^a}{2\pi} \frac{x^j}{r^2}\,. \tag{13}$$

With this solution, the constraints $\chi_i^A = 0$, $A = 1, 2$ eqs. (10) are read as

$$\begin{cases} \widehat{\pi}_{1i} = \widehat{J}_{1a}\,\omega_i{}^a(\boldsymbol{\xi}_1) + \widehat{p}_{1a}\, e_i{}^a(\boldsymbol{\xi}_1) = \varepsilon_{ij} \dfrac{(\xi_1^j - \xi_2^j)}{(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)^2} \dfrac{1}{2\pi} [\widehat{J}_1 \cdot \widehat{p}_2 + (\widehat{J}_2 - \alpha \widehat{p}_2) \cdot \widehat{p}_1]\,, \\ \widehat{\pi}_{2i} = \widehat{J}_{2a}\,\omega_i{}^a(\boldsymbol{\xi}_2) + \widehat{p}_{2a}\, e_i{}^a(\boldsymbol{\xi}_2) = \varepsilon_{ij} \dfrac{(\xi_2^j - \xi_1^j)}{(\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1)^2} \dfrac{1}{2\pi} [(\widehat{J}_1 - \alpha \widehat{p}_1) \cdot \widehat{p}_2 + \widehat{J}_2 \cdot \widehat{p}_1]\,, \end{cases} \tag{14}$$

and the braiding operator can be evaluated to be

$$\begin{aligned} \exp\left[i \oint A_\mu \,\mathrm{d}\xi^\mu\right] &= \exp\left[i \oint \widehat{\pi}_{2i}\,\mathrm{d}\xi_2^i\right] \\ &= \exp\left[i \oint (\widehat{J}_{2a}\,\omega_i^a(\boldsymbol{\xi}_2) + \widehat{p}_{2a}\, e_i^a(\boldsymbol{\xi}_2))\,\mathrm{d}\xi_2^i\right] \\ &= \exp\left[i(\widehat{J}_1 \cdot \widehat{p}_2 + \widehat{J}_2 \cdot \widehat{p}_1 - \alpha \widehat{p}_1 \cdot \widehat{p}_2)\right]. \end{aligned} \tag{15}$$

We now consider the scattering process of the above situation in the test-particle approximation, by using the phase space path integral formalism. Since we can solve the constraints explicitly in this case, let us assume all the constraints are imposed. Then the Lagrangian is reduced to

$$L = (p_{\bar{i}}\, \dot{q}^{\bar{i}} + \bar{s} \cosh\theta\, \dot{\phi}) + (J_a\, \omega_i{}^a + p_a\, e_i{}^a)\, \dot{\xi}^i - \sqrt{\vec{p}^{\,2} + m^2} \tag{16}$$

up to constant term. Notice that we dropped the index of the particle for simplicity. To get the idea on what our strategy is, let us consider a simple case in which both particles are spinless. Then, in non-relativistic limit, with the gauge choice $\xi^\mu = \delta_a^\mu q^a$, the scattering amplitude becomes

$$S = \int [\mathrm{d}\vec{q}][\mathrm{d}\vec{p}] \exp\left[i \int \mathrm{d}t \left(\vec{p} \cdot \dot{\vec{q}} - \frac{\vec{p}^{\,2}}{2m} - lM \frac{\dot{\theta}}{2\pi}\right)\right], \tag{17}$$

where $l = \varepsilon_{ij} q^i p^j = \vec{q} \times \vec{p}$, θ is the angular coordinate of q^i, and M and m are the masses of the source and the test particle, respectively. Then, after completing the integral over the momentum $\vec{p}$, we at last obtain the expression for the propagator as

$$G(\vec{r}_f, t_f; \vec{r}_i, t_i) = \int [\mathrm{d}\vec{q}] \exp\left[i \int \mathrm{d}t \frac{m}{2} \left[\dot{\vec{q}}^2 + \vec{q}^2 \left(1 - \frac{M}{2\pi}\right)^2 \dot{\theta}^2 \right]\right]. \tag{18}$$

Notice the appearance of the deficit factor before the θ-term. Therefore, the propagator gives exactly the same result as that in [14, 15], which was demonstrated before in [4]. The important point here is that we started just with the *flat* coordinate $\vec{q}$, but we arrived at the same result as that obtained by the conventional approach which uses from the beginning the *geometric* coordinate which manifests a deficit angle. More details, in addition to the generalization to the cases with an arbitrary number of particles and/or with arbitrary spins, will be discussed elsewhere [16].

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REFERENCES

[1] ACHUĆARRO A. and TOWNSEND P. K., *Phys. Lett. B*, **180** (1986) 85.
[2] WITTEN E., *Nucl. Phys. B*, **311** (1988) 46.
[3] CHO J. H. and LEE H. J., SNUTP preprint (1995).
[4] DE SOUSA GERBERT P., *Nucl. Phys. B*, **346** (1990) 440.
[5] SKAGERSTAM B. S. and STERN A., *Int. J. Mod. Phys. A*, **5** (1990) 1575.
[6] FADDEEV L. and JACKIW R., *Phys. Rev. Lett.*, **60** (1988) 1692.
[7] GRIGNANI G. and NARDELLI G., *Phys. Lett. B*, **264** (1991) 45; *Nucl. Phys. B*, **370** (1992) 491; *Phys. Rev. D*, **45** (1992) 2719.
[8] KABAT D. and ORTIZ M. E., *Phys. Rev. D*, **49** (1994) R1684.
[9] LEE T. and ZOH S., *J. Kor. Phys. Soc.*, **28** (1995) S539.
[10] PLYUSHCHAY M. S., *Nucl. Phys. B*, **362** (1991) 54; *Phys. Lett. B*, **262** (1991) 71; *Int. J. Mod. Phys. A*, **7** (1992) 7045; FORTE S. and JOLICAEUR T., *Nucl. Phys. B*, **350** (1991) 589; CHAICHIAN M., FELIPE R. G. and MARTINEZ D. L., *Phys. Rev. Lett.*, **71** (1993) 3045; JACKIW R. and NAIR V. P., MIT preprint (1993); LEE T., *J. Kor. Phys. Soc.*, **28** (1995) 400.
[11] VERLINDE E., in *Modern Quantum Field Theory* (World Scientific, Singapore) 1991; LO H.-K. and PRESKILL J., *Phys. Rev. D*, **48** (1993) 4821.
[12] LEE T. and OH P., *Phys. Lett. B*, **319** (1993) 497; *Ann. Phys.* (*N.Y.*), **235** (1994) 413; *Phys. Rev. Lett.*, **72** (1994) 1141.
[13] BAK D., JACKIW R. and PI S.-Y., *Phys. Rev. D*, **49** (1994) 6778.
[14] 'T HOOFT G., *Commun. Math. Phys.*, **117** (1988) 685.
[15] DESER S. and JACKIW R., *Commun. Math. Phys.*, **118** (1988) 495.
[16] LEE T. and ZOH S., in preparation.