Evolution of textures (*)

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Summary. — Textures are the objects that can be produced during the phase transition where the original symmetry G is broken to the remaining symmetry H and the homotopy group $\pi_3(G/H)$ is nontrivial. We review the properties of the static texture configurations both in flat space-time and in the fully general relativistic case. They are generically unstable. We then investigate the time evolution of spherically symmetric textures in flat space-time. Our numerical results confirm the existence of the critical winding number for which it takes infinite time for the texture to unwind. When the initial winding number is greater than the critical value, textures collapse until the unwinding takes place and, after the unwinding, exhibit no trend of rewinding but expand with the energy being dispersed.

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During the symmetry breakings in the early universe, various kinds of topological defects could have been produced. Suppose the original symmetry G is broken with the remaining symmetry being H, then topological defects can be produced if the homotopy group $\pi_n(G/H)$ is nontrivial. Cosmic strings are produced in the case where $n=1$ and monopoles are produced when $n=2$. When $n=3$ we can have textures [1] which are of interest to us here.

The simplest model that can give rise to textures is given by the Lagrangian density

$$\mathscr{L} = \frac{1}{2}\partial^\mu \Phi^a \partial_\mu \Phi^a - \lambda(|\Phi|^2 - v^2)^2, \tag{1}$$

where $\Phi^a (a=0, 1, 2, 3)$ is a 4-component real scalar field. The original global $O(4)$ symmetry is broken when the vacuum expectation value of the square of the scalar field

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assumes the value v^2. The remaining symmetry is $O(3)$ and we have 3 massless modes and 1 massive mode with $m^2 = 4\lambda v^2$. The homotopy group $\pi_3(O(4)/O(3))$ is nontrivial and, as the scalar field at each space point can assume a random value in the degenerate vacuum manifold, textures can form. Unlike monopoles and cosmic strings for which some portion of space has to remain in the false vacuum state, textures do not need a false vacuum region, *i.e.* the values of the scalar field for the texture configuration are those in the true vacuum manifold at all space points and the only energy in the system is the gradient energy of the scalar field.

If one tries to extend this model to the case with the local symmetry, one does not come up with any interesting situation as far as topological defects are concerned. The value of the scalar field at each space point can be represented as the result of a local gauge transformation from some fixed value in the vacuum manifold,

$$\Phi(x) = g(x)\,\Phi_0. \tag{2}$$

Then, by introducing the gauge field in the appropriate pure gauge form,

$$A_\mu = -\frac{i}{e}(\partial_\mu g)\,g^{-1}, \tag{3}$$

all the gradient energy can be erased and one ends up with merely another vacuum configuration.

Another important feature in the case of textures is that there is no finite-energy stable solution to the texture field equation as can be shown easily by the use of Derrick's theorem [2]. The system evolves with time and the scalar field may get away from the vacuum manifold.

If the scalar field is kept within the vacuum manifold, one can have a conserved current defined by

$$j^\mu = \frac{1}{12\pi^2 v^4}\,\varepsilon^{\mu\nu\varrho\lambda}\,\varepsilon^{abcd}\,\Phi^a\,\partial_\nu\Phi^b\,\partial_\varrho\Phi^c\,\partial_\lambda\Phi^d. \tag{4}$$

The total charge is given by

$$Q = \int \mathrm{d}^3x\, j^0 \tag{5}$$

which also corresponds to the winding number of the scalar field around the vacuum manifold. If the scalar field assumes the form of the spherically symmetric ansatz,

$$\Phi^a = v\begin{pmatrix} \sin F(t,\,r)\,\sin\theta\sin\phi \\ \sin F(t,\,r)\,\sin\theta\cos\phi \\ \sin F(t,\,r)\,\cos\theta \\ \cos F(t,\,r) \end{pmatrix}, \tag{6}$$

the winding number can be read off the values of F at the origin and infinity. For example, $Q = -1$ when $F(r=\infty) = \pi$ and $F(r=0) = 0$. As mentioned before, the scalar field may get away from the vacuum manifold as the system evolves with time, and then the winding number is not conserved. We will discuss the process of unwinding of the texture configuration later.

The field equation obtained by extremizing the action is

$$\partial_\mu \partial^\mu \Phi^a + 4\lambda(|\Phi|^2 - v^2)\,\Phi^a = 0\,. \tag{7}$$

For the case where the scalar field remains very close to the vacuum manifold, we may reduce the field equation to the form of the nonlinear σ-model approximation

$$\partial_\mu \partial^\mu \Phi^a = -\frac{\partial_\mu \Phi^b \partial^\mu \Phi^b}{v^2}\,\Phi^a, \tag{8}$$

$$\Phi^a \Phi^a = v^2. \tag{9}$$

With the spherically symmetric ansatz eq. (6), one is left with one scalar field F and the field equation for F is

$$\frac{\partial^2 F}{\partial t^2} - \frac{\partial^2 F}{\partial r^2} - \frac{2}{r}\frac{\partial F}{\partial r} = -\frac{1}{r^2}\sin 2F\,. \tag{10}$$

Turok and Spergel [3] showed an exact analytic solution to the above equation which is in the self-similar form,

$$F = F\left(\frac{r}{t}\right) = \begin{cases} 2\arctan\left(-\dfrac{r}{t}\right), & t<0\,, \\ 2\arctan\left(\dfrac{r}{t}\right)+\pi\,, & t>0\,,\quad r<t\,, \\ 2\arctan\left(\dfrac{t}{r}\right)+\pi\,, & t>0,\quad r>t\,. \end{cases} \tag{11}$$

This solution represents the case where the winding number changes from -1 ($t<0$) to 0 ($t>0$).

One can also consider static solutions even though they are unstable. The field equation becomes

$$\frac{\mathrm{d}^2 F}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}F}{\mathrm{d}r} = \frac{1}{r^2}\sin 2F\,. \tag{12}$$

Iwasaki and Ohyama [4] showed, by numerical calculations, that the static solution to this equation is unique and correspons to the winding number $1/2$. It is unique in the sense that the only other allowed solutions are those obtained through the transformations

$$\begin{aligned} F(r) &\to F(\lambda r) && (\lambda = \text{const})\,, \\ F(r) &\to F(r) + n\pi && (n = \text{integer})\,, \\ F(r) &\to -F(r)\,. \end{aligned} \tag{13}$$

Aminneborg [5] considered the perturbations of the form

$$F(t_0,\, r) = aF(r)\,, \tag{14}$$

$$\dot{F}(t_0, r) = 0 \tag{15}$$

and showed that the configuration collapses (expands) when $a > 1$ corresponding to the initial winding number greater (smaller) than $1/2$. Although this cannot be the general proof as it considers only a certain class of perturbations, it suggests a special role of the winding number $1/2$ in the evolution of textures.

Next we consider the case where the texture field is coupled with gravity. With the use of the spherically symmetric ansatz for the space-time metric

$$\mathrm{d}s^2 = e^{2q(r)}\,\mathrm{d}t^2 - e^{2p(r)}\,\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{16}$$

and for the scalar field (eq. (6) with F being dependent on r only in this case), the fully coupled Einstein and texture field equations become

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{1}{2r}(1 - e^{2p}) + 2\pi G v^2 r\left(\left(\frac{\mathrm{d}F}{\mathrm{d}r}\right)^2 + e^{2p}\,\frac{2\sin^2 F}{r^2}\right), \tag{17}$$

$$\frac{\mathrm{d}q}{\mathrm{d}r} = -\frac{1}{2r}(1 - e^{2p}) + 2\pi G v^2 r\left(\left(\frac{\mathrm{d}F}{\mathrm{d}r}\right)^2 - e^{2p}\,\frac{2\sin^2 F}{r^2}\right), \tag{18}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}F}{\mathrm{d}r}\right) + \left(\frac{\mathrm{d}q}{\mathrm{d}r} - \frac{\mathrm{d}p}{\mathrm{d}r}\right) r^2\,\frac{\mathrm{d}F}{\mathrm{d}r} - e^{2p}\sin(2F) = 0\,. \tag{19}$$

We found [6], by numerically integrating the above equations, that the static solution is unique again in this case and it corresponds to winding number $1/2$. No other static solutions are allowed except those that can be obtained through the transformations

$$\begin{cases} F(r),\, p(r) \;\rightarrow\; F(\lambda r),\, p(\lambda r) & (\lambda = \text{const}) \\ F(r),\, p(r) \;\rightarrow\; F(r) + n\pi,\, p(r) & (n = \text{integer}) \\ F(r),\, p(r) \;\rightarrow\; -F(r),\, p(r)\,. & \end{cases} \tag{20}$$

The geometry of the spatial hypersurface in the asymptotic region of $r \rightarrow \infty$ is found to be that of a flat space with a deficit solid angle of the amount

$$\Delta\Omega = 32\pi^2 G v^2. \tag{21}$$

Now let us consider the time evolution of textures. We go back to the flat space-time, but still assume the spherical symmetry. We allow the scalar field to get away from the vacuum manifold and therefore introduce two functions R and Z instead of just one:

$$\Phi^a = \begin{pmatrix} R(t, r)\sin\theta\sin\phi \\ R(t, r)\sin\theta\cos\phi \\ R(t, r)\cos\theta \\ Z(t, r) \end{pmatrix}. \tag{22}$$

With this ansatz, the field equations become

$$(23)\qquad \begin{cases} \ddot{R} - R'' - \dfrac{2R'}{r} + \dfrac{2R}{r^2} = -4\lambda(R^2 + Z^2 - \eta^2)\, R\,, \\ \ddot{Z} - Z'' - \dfrac{2Z'}{r} = -4\lambda(R^2 + Z^2 - \eta^2)\, Z\,, \end{cases}$$

where the primes and overdots represent derivatives with respect to r and t, respectively. Since these are second-order partial differential equations, one can be free to specify the initial conditions for R, Z, $\dot{R}$, and $\dot{Z}$ with some boundary conditions. For regular solutions, $R''|_{r=0}$ and $Z'|_{r=0}$ should be zero, and for the spherical symmetry, $R(t, 0)$ should be zero. For numerical work, we make all the variables dimensionless by performing the scale transformations,

$$R \to \eta R\,, \qquad Z \to \eta Z\,, \qquad \text{and} \qquad t \to \frac{t}{2\eta\sqrt{\lambda}}\,, \qquad r \to \frac{r}{2\eta\sqrt{\lambda}}\,.$$

We start with the initial configuration

$$(24)\qquad \begin{cases} R(0, r) = \sin\chi(0, r)\,, \quad \dot{R} = 0\,, \\ Z(0, r) = \cos\chi(0, r)\,, \quad \dot{Z} = 0\,, \end{cases}$$

where $\chi(0, r) = q\tan^{-1}(r/t_0)$. This represents the texture field initially at rest on the vacuum manifold with a configuration similar to the self-similar solution in the nonlinear σ-model approximation. It contains two parameters q and t_0. By varying the value of q we vary the initial winding number, with $q = 2$ corresponding to $Q = 1$, and t_0 determines the initial spatial size of the texture. Now we sketch the results of our numerical calculations [7].

For the case where the initial winding number is 1, for example, the value of Z at the origin ($r = o$) starts with 1 at $t = 0$, fluctuates around the initial value 1 for a while, drops quickly passing through zero at a certain moment and then eventually oscillates around -1. The moment when the value of Z passes through zero corresponds to the moment when the unwinding occurs. It does not show the tendency to rewind. The energy density at the origin goes up near the time of unwinding and then drops again. We next investigated how the time it takes for the texture to unwind depends on the initial winding number. Our numerical results show that, as the value of the initial winding is reduced, it takes longer for the texture to unwind. There appears no unwinding as the value of the initial winding approaches somewhere around 0.6.

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