

Thermodynamics of black holes in Brans-Dicke gravity (*)

H. KIM and Y. KIM

Department of Physics, Sogang University - C.P.O. Box 1142, Seoul 100-611, Korea

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Summary. — It has recently been argued that non-trivial Brans-Dicke black-hole solutions different from the usual Schwarzschild solution could exist. We attempt here to “censor” these non-trivial Brans-Dicke black-hole solutions by examining their thermodynamic properties. Quantities like Hawking temperature and entropy of the black holes are computed. The analysis of the behaviors of these thermodynamic quantities appears to show that even in Brans-Dicke gravity, the usual Schwarzschild space-time turns out to be the only physically relevant uncharged black-hole solution.

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1. – Introduction

Brans-Dicke (BD) gravity [1] is perhaps the most well-known alternative theory of classical gravity to Einstein's general relativity. This theory can be regarded as an economic modification of general relativity which accommodates both Mach's principle and Dirac's large-number hypothesis as new ingredients. Ever since it first appeared, it has remained as a viable theory of classical gravity in that it passed all the available observational/experimental tests provided a certain restriction on the generic parameter “ ω ” of the theory is imposed [2]. Shortly after the appearance of their first work [1], one of the authors, Brans, provided static, spherically-symmetric metric and scalar-field solutions to the vacuum BD field equations [3]. Since the gravitational collapse and the subsequent black-hole formation is generally of great interest in classical gravity, in the present work we would like to address questions like: under what circumstances Brans' solutions can describe black-hole space-times and if they actually do, could they really be non-trivial ones different from the general relativistic

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black holes? It will then be discussed that although non-trivial BD black-hole solutions different from the usual Schwarzschild solution appear to exist as suggested recently by Campanelli and Lousto (CL) [4], when “censored” by quantum aspects of black holes, namely their thermodynamics, it can be shown that they cannot really arise in nature. Brans [3] actually has provided an exact static and isotropic solution to the vacuum BD field equations in four possible forms depending on the values of the arbitrary constants appearing in the solution. In the present work, we consider only the Brans “type I” solution since it is the only form that is permitted for all values of the “BD parameter”, ω (the other three forms are allowed only for negative values of ω , $\omega \leq -3/2$). In fact, in this work we will exclusively assume that the parameter ω is *positive* since it has been prescribed so originally in the BD theory itself [1] (namely according to Brans and Dicke, the positive contribution of nearby matter to the space-time-dependent Newton’s constant demands ω to be positive) and it also has been constrained by experiments [2] (for $\omega \geq 500$, the theory is in reasonable accord with all available experiments thus far). In fact, there is another crucial reason why ω has to be positive from field theory’s viewpoint. Namely, in order for the BD scalar field Φ to have “canonical (positive-definite)” kinetic energy, ω needs to be positive.

The vacuum BD gravity is described by the action (we shall work in the Misner-Thorne-Wheeler sign convention)

$$(1) \quad S = \frac{1}{16\pi} \int d^4x \sqrt{g} \left[\Phi R - \omega g^{\mu\nu} \frac{\nabla_\mu \Phi \nabla_\nu \Phi}{\Phi} \right]$$

and the field equations

$$(2) \quad \begin{cases} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{\omega}{\Phi^2} \left[\nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \Phi \nabla^\alpha \Phi \right] + \frac{1}{\Phi} [\nabla_\mu \nabla_\nu \Phi - g_{\mu\nu} \nabla_\alpha \nabla^\alpha \Phi], \\ \nabla_\alpha \nabla^\alpha \Phi = 0. \end{cases}$$

The Brans type-I solution obtained in isotropic coordinates is given by [2]

$$(3) \quad \begin{cases} ds^2 = - \left(\frac{\tilde{r} - r_0}{\tilde{r} + r_0} \right)^{2(Q-\chi)} dt^2 + \left(1 + \frac{r_0}{\tilde{r}} \right)^4 \left(\frac{\tilde{r} - r_0}{\tilde{r} + r_0} \right)^{2(1-Q)} [d\tilde{r}^2 + \tilde{r}^2 d\Omega_2^2], \\ \Phi(\tilde{r}) = \left(\frac{\tilde{r} - r_0}{\tilde{r} + r_0} \right)^\chi, \end{cases}$$

with

$$(4) \quad Q^2 + \left(1 + \frac{\omega}{2} \right) \chi^2 - Q\chi - 1 = 0.$$

Here, the arbitrary constants Q , χ appearing in the solution are subject to the constraint in eq. (4) and are related to Brans’ original notation [3] by $Q = (1 + c)/\lambda$, $\chi = c/\lambda$ and to the notation of CL by $Q = (1 - n)$, $\chi = -(m + n)$. Next, $\Phi(\tilde{r})$ denotes the “Brans-Dicke scalar field” and the quantity r_0 , which is related to Brans’ notation [3] by $r_0 = B$ (or by $r_0 = \tilde{r}_0$ to the notation of CL), is a “mass parameter” related to “scalar” and “tensor” mass by $M_s = -\chi r_0$, $M_t = (2Q - \chi) r_0$, respectively. The quantities M_s and M_t are constants and related to the “Keplerian” (active gravitational) mass M

measured by a test particle by $M = M_t + M_s$. Like the ADM mass in general relativity, the tensor mass M_t is positive-definite and decreases monotonically by emitting gravitational radiation. The scalar mass M_s and the Kepler mass M , however, share none of these properties [5]. Also note that for parameter values $Q = 1$, $\chi = 0$ (or $m = n = 0$ in the notation of CL), the above exact solution in eq. (3) reduces to the usual Schwarzschild solution in Einstein gravity implying that it indeed is a particular solution of BD field equations.

Now, consider the coordinate transformation to the standard Schwarzschild coordinates

$$r = \tilde{r} \left(1 + \frac{r_0}{\tilde{r}} \right)^2 \quad \text{or} \quad \tilde{r} = \frac{1}{2} [(r - 2r_0) + (r^2 - 4r_0 r)^{1/2}].$$

In terms of Schwarzschild coordinates, then, the Brans type-I solution in eq. (3) now takes the form

$$(5) \quad \begin{cases} ds^2 = - \left(1 - \frac{4r_0}{r} \right)^{(Q-\chi)} dt^2 + \frac{1}{(1 - 4r_0/r)^Q} dr^2 + \frac{r^2}{(1 - 4r_0/r)^{(Q-1)}} d\Omega_2^2, \\ \Phi(r) = \left(1 - \frac{4r_0}{r} \right)^{\chi/2}. \end{cases}$$

2. - Non-trivial classical BD black-hole solutions

As mentioned earlier, CL recently examined this vacuum solution and claimed that under certain circumstances it could represent non-trivial BD black-hole solutions different from the usual Schwarzschild solution. In what follows, we first give a brief review of their argument. Obviously, in order for this metric solution to represent a black-hole space-time, the parameters (Q, χ) appearing in the solution should satisfy certain conditions. One straightforward way of obtaining such conditions is to consider under what circumstances an event horizon forms. To do so, CL studied outgoing null geodesics and looked for a condition under which the surface at $r = 4r_0$ (or in isotropic coordinates, at $\tilde{r} = r_0$) could behave as an event horizon. It turned out that it happens provided

$$2Q - \chi > 1.$$

A simple argument that can lead to this condition goes as follows; first, since the metric solution of the vacuum BD field equations above has time translational ($t \rightarrow t + \delta t$) isometry, a timelike Killing field $\xi^\mu = \delta_0^\mu$ exists correspondingly. Now, as is well known, in general if ξ^μ is a hypersurface orthogonal Killing field which commutes with all others, then the surfaces where $\xi^\mu \xi_\mu = 0$ are ‘‘Killing horizons’’. Thus for the case at hand, in order to see if a Killing horizon such as an event horizon develops, we need to find out under what condition $\xi^\mu \xi_\mu$ can have zeros. It is then straightforward to see that $\xi^\mu \xi_\mu = g_{00} = (1 - 4r_0/r)^{(Q-\chi)} = 0$ can have a zero at $r = 4r_0$ if $(Q - \chi) > 0$. In addition, we do not want to have a curvature singularity at the surface $r = 4r_0$ which is a candidate for an event horizon. Thus from $g_{\phi\phi} = r^2 (1 - 4r_0/r)^{(1-Q)} \sin^2 \theta$, we demand $(Q - 1) > 1$. These two conditions, when properly put together, then yield the above condition $2Q - \chi > 1$ which

may be thought of as the condition for this metric solution in eq. (5) to represent possibly a black-hole space-time.

Next, in order for this metric solution to represent truly a black-hole space-time, it should have a “regular” event horizon. As briefly mentioned above, then, we should further require that the curvature, for instance, have a non-singular behavior at the null surface $r = 4r_0$. Thus in this time we consider the Kretschmann curvature invariant

$$(6) \quad I = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \frac{(4r_0)^2}{r^6} \left(1 - \frac{4r_0}{r}\right)^{(2Q-4)} \left\{ \left(\frac{2r_0}{r}\right)^2 I_1(Q, \chi) + 4 \left(\frac{2r_0}{r}\right) I_2(Q, \chi) + 6 I_3(Q, \chi) \right\},$$

where

$$\begin{aligned} I_1(Q, \chi) &= 7Q^4 + 16Q^3 + 14Q^2 + 8Q + \chi^4 + 2\chi^3 + 6\chi^2 - 16Q^3\chi + \\ &\quad + 15Q^2\chi^2 - 6Q\chi^3 - 51Q^2\chi + 20Q\chi^2 - 12Q\chi + 3, \\ I_2(Q, \chi) &= -4Q^3 - 6Q^2 + 11Q + \chi^3 - 3\chi^2 + 7Q^2\chi - 5Q\chi^2 + 6Q\chi - 13, \\ I_3(Q, \chi) &= 2Q^2 + \chi^2 - 2Q\chi. \end{aligned}$$

Note first that this curvature invariant goes to zero as $r \rightarrow \infty$ as fast as $I \rightarrow O(r^{-6})$ and for the special case of interest, $Q = 1$, $\chi = 0$, *i.e.* for the Schwarzschild solution, it reduces to $I = (48(2r_0)^2)/r^6 = (48M^2)/r^6$ (where $M = 2r_0$ is the ADM mass for Schwarzschild solution) as it should. Next, it is easy to see that the condition for non-singular behavior of the curvature invariant at $r = 4r_0$ amounts to the constraint

$$Q \geq 2.$$

Finally, put them altogether, the condition for the metric solution in eq. (5) to represent a black-hole space-time with regular event horizon turns out to be

$$(7) \quad 2Q - \chi > 1 \quad \text{and} \quad Q \geq 2.$$

Therefore it appears that for these values of the parameters appearing in the solution, the metric in eq. (5) could represent a non-trivial black-hole space-time different from the usual Schwarzschild solution in general relativity. At this point, let us recall the well-known Hawking’s theorem on black holes in BD theory [6]. Long ago, Hawking put forward a theorem which states that stationary black holes in BD theory are identical to those in general relativity. To be more concrete, Hawking extended some of his theorems for general relativistic black holes to BD theory and showed that any object collapsing to a black hole in BD gravity must settle into final equilibrium state which is either Schwarzschild or Kerr space-time. And in doing so, he assumed that the BD scalar field Φ satisfies the weak-energy condition and is constant outside the black hole. Now one may be puzzled. It is being claimed that non-trivial BD black-hole solutions different from the Schwarzschild black hole might exist in apparent contradiction to Hawking’s theorem. The possible existence of non-trivial BD black-hole solutions discussed here, however, does not really contradict Hawking’s theorem since they violate the weak-energy condition on the BD scalar field Φ that we now show below.

The energy-momentum tensor for the BD scalar field is given by

$$(8) \quad T_{\mu\nu}(\Phi) = \frac{1}{8\pi} \left[\frac{\omega}{\Phi^2} \left(\nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \Phi \nabla^\alpha \Phi \right) + \frac{1}{\Phi} \nabla_\mu \nabla_\nu \Phi \right]$$

from which one can readily compute

$$(9) \quad T_{\mu\nu}(\Phi) \xi^\mu \xi^\nu = T_{00} = -\frac{1}{8\pi} (Q^2 - 1) \left(\frac{2r_0}{r} \right)^2 \left(1 - \frac{4r_0}{r} \right)^{2(Q-1)-\chi}.$$

Thus for $Q \geq 2$, which is the condition satisfied by regular BD black-hole solutions, $T_{\mu\nu}(\Phi) \xi^\mu \xi^\nu < 0$, namely the weak-energy condition on the BD scalar field is violated. And this is the only means for the non-trivial BD black-hole solutions to evade Hawking’s theorem. Since they violate the weak-energy condition on BD scalar field, one may simply reject them as unphysical solutions. One may, however, still take them seriously as CL did by keeping the viewpoint that demanding the weak-energy condition on the BD scalar field is not absolutely compelling. In the present work, we choose to take the viewpoint of the latter and then proceed further. In the next section, we shall attempt to “censor” these non-trivial BD black-hole solutions by examining their thermodynamic properties. Our philosophy here is that not all classically allowed black holes might be truly realistic and they should be further censored by quantum aspects of black holes, *i.e.* their thermodynamics.

3. – Thermodynamics of the non-trivial BD black holes

We now turn to the investigation of the thermodynamics of non-trivial BD black-hole solutions discussed thus far. The long-standing gap between thermodynamics and gravity has been essentially bridged by Hawking and later by many authors [7] who first have shown, via the study of quantum fields propagating on black-hole background space-times, that black holes do evaporate as if they were black bodies. Namely the black-hole thermodynamics is essentially associated with the quantum aspect of the black-hole physics. Therefore, one may expect that by examining thermodynamic properties of a black hole, one can, to some extent, explore its quantum aspect. Besides, since the practical study of black-hole thermodynamics begins and ends with the evaluation of black hole’s temperature and entropy, we shall attempt to compute them. First, in order to obtain the Hawking temperature T_H measured by an observer in the asymptotic region, one needs to compute the surface gravity κ of the black hole and relate it to the temperature by $T_H = \kappa/2\pi$ [7]. Although the identification of the black-hole temperature with $T_H = \kappa/2\pi$ was originally derived from studies of linear, free quantized fields propagating in given black-hole geometries [7], it holds equally well for interacting fields of arbitrary spin and for general black-hole space-times [8]. Thus our task reduces to the calculation of the surface gravity. In physical terms, the surface gravity κ is the force that must be exerted to hold a unit test mass at the horizon and it is given in a simple formula as [9]

$$(10) \quad \kappa^2 = -\frac{1}{2} (\nabla^\mu \chi^\nu) (\nabla_\mu \chi_\nu) = -\frac{1}{2} g^{\alpha\beta} g_{\mu\nu} \Gamma_{\alpha\varrho}^\mu \chi^\varrho \Gamma_{\beta\lambda}^\nu \chi^\lambda,$$

where χ^μ is a Killing field of the given stationary black hole which is normal to the horizon and here the evaluation on the horizon is assumed. For our static BD black hole given in eq. (5), χ^μ is just the timelike Killing field, $\chi^\mu = \xi^\mu$ and hence the surface gravity and the Hawking temperature are computed to be

$$(11) \quad T_H = \frac{\kappa}{2\pi} = \frac{1}{2\pi} (Q - \chi) \left(\frac{2r_0}{r^2} \right) \left(1 - \frac{4r_0}{r} \right)^{(Q - \chi/2 - 1)} \Big|_{r=4r_0}.$$

Having established the expression for the Hawking temperature T_H of the black hole, we next go on to find the ‘‘local temperature’’. Generally, the local (or ‘‘Tolman red-shifted’’) temperature $T(r)$ of an accelerated observer can be obtained by blue-shifting the Hawking temperature T_H of an observer in the asymptotic region from infinity to a finite point r . In other words, according to ‘‘Tolman relation’’ [10], the local temperature $T(r)$ measured by a moving, accelerated detector is related to the Hawking temperature T_H measured by a detector in the asymptotic region by

$$(12) \quad T(r) = (-g_{00})^{-1/2} T_H.$$

Thus the local temperature of our BD black hole is found to be

$$(13) \quad T(r) = \frac{1}{2\pi} (Q - \chi) \left(\frac{2r_0}{r^2} \right) \left(1 - \frac{4r_0}{r} \right)^{(Q - \chi/2 - 1)} \Big|_{r=4r_0} \times \left(1 - \frac{4r_0}{r} \right)^{-(1/2)(Q - \chi)}$$

which behaves asymptotically as $T(r \rightarrow \infty) \rightarrow T_H$ as it should. Next we turn to the computation of the black hole’s entropy. Generally speaking, the black-hole entropy can be evaluated in three ways; firstly, following the Bekenstein-Hawking proposal [11], one can argue *a priori* that the entropy of a black hole must be proportional to the surface area of its event horizon (*i.e.* $S = 1/4A$). Alternatively, knowing the Hawking temperature T_H and chemical potentials (*i.e.* Coulomb potential at the event horizon Φ_H for the conserved $U(1)$ charge Q and angular velocity of the horizon Ω_H for the conserved angular momentum J generally), one may integrate the first law of black-hole thermodynamics $T_H dS = dM - \Phi_H dQ - \Omega_H dJ$ to obtain the entropy [8]. Thirdly, according to Gibbons and Hawking [8], thermodynamic functions including the black-hole entropy can be computed directly from the saddle point approximation to the gravitational partition function (namely the generating functional analytically continued to the Euclidean space-time). In the present case where we consider the static, isotropic black holes in BD gravity theory, the expression for the Hawking temperature given in eq. (11) renders it awkward to employ the second method which uses the first law of black-hole thermodynamics. Nor can we naively adopt the Bekenstein-Hawking relation $S = 1/4A$ to obtain black-hole entropy. In fact, the relation $S = 1/4A$ has been established essentially in the context of the Einstein gravity via the first law of black-hole thermodynamics which has been derived from the expression for the Bondi mass of a stationary black hole [9]. In the context of BD gravity, due to the addition of the gravitational scalar degree of freedom (*i.e.* the BD scalar field Φ), the expression for the mass of a hole is subject to a modification which, in turn, would result in another modification in the first law of thermodynamics violating the exact relation, $S = 1/4A$. Therefore, here, as a reliable way of evaluating the entropy, we should resort to the method suggested by Gibbons and Hawking [8] since it is based on the fundamental definition of entropy. To illustrate the procedure

briefly, we begin with the vacuum Euclidean BD gravity action and field equations

$$(14) \quad I[g, \Phi] = -\frac{1}{16\pi} \left[\int_Y d^4x \sqrt{g} \left(\Phi R - \omega g^{\mu\nu} \frac{\nabla_\mu \Phi \nabla_\nu \Phi}{\Phi} \right) + 2 \int_{\partial Y} d^3x \sqrt{h} \Phi (K - K_0) \right].$$

The boundary term in the Euclidean BD action $I[g, \Phi]$ above has been determined as follows: following Gibbons and Hawking [8], we start with the form $\int_{\partial Y} d^3x \sqrt{h} B$, where

$B = -(1/8\pi) \Phi K + C$. Now for asymptotically flat space-times where the boundary ∂Y of the 4-dimensional manifold Y can be taken to be the product of the (analytically continued) time axis with a 2-sphere of large radius, *i.e.* $\partial Y = S^1 \times S^2$, it is natural to choose C so that B vanishes for the flat space-time metric $\eta_{\mu\nu}$. Thus we have $B = -(1/8\pi) \Phi (K - K_0)$, where K and K_0 are traces of the second fundamental form of ∂Y in the metric $g_{\mu\nu}$ and $\eta_{\mu\nu}$, respectively (one may choose to take $B = -(1/8\pi) \cdot (\Phi K - \Phi_0 K_0)$ where $\Phi_0 = \text{const} = 1$. The choice of the latter, however, leads to no substantial change in the conclusion that we shall draw from the former). Now the gravitational partition function and the entropy are given in the saddle point approximation by

$$(15) \quad \begin{cases} Z = \int [dg_{\mu\nu}] [d\Phi] e^{-I[g, \Phi]} \simeq e^{-I[g^c, \Phi^c]}, \\ S = \ln Z + \beta M \simeq -I[g^c, \Phi^c] + \frac{M}{T_H}, \end{cases}$$

where the superscript “ c ” denotes the saddle point of the action. Since the saddle point of the action is nothing but the solution to the vacuum BD field equations in eq. (2), first we have

$$\int_Y d^4x \sqrt{g} \left(\Phi R - \omega g^{\mu\nu} \frac{\nabla_\mu \Phi \nabla_\nu \Phi}{\Phi} \right) = 0$$

at (g^c, Φ^c) and thus the non-vanishing contribution to $I[g^c, \Phi^c]$ comes only from the boundary term on ∂Y . Here the computation of the term $\int_{\partial Y} d^3x \sqrt{h} \Phi K_0$ is straight-

forward since we know that for flat space-time, $K_0 = 2/r$. Next from the definition of the second fundamental form K [9] and from the spherical symmetry of the BD black-hole solution given in eq. (5), we have $\int_{\partial Y} d^3x \sqrt{h} \Phi K = \Phi(\partial/\partial n) \int_{\partial Y} d^3x \sqrt{h}$, where

$(\partial/\partial n) \int_{\partial Y} d^3x \sqrt{h}$ is the derivative of the area $\int_{\partial Y} d^3x \sqrt{h}$ of ∂Y as each point of ∂Y is moved an equal distance along the outward unit normal n . Then the result of the actual calculation is

$$I[g^c, \Phi^c] = 2\pi r_0 \kappa^{-1} (3Q + \chi - 2) + O(r_0^2 r^{-1}).$$

Therefore, the black-hole entropy is found to be

$$(16) \quad S = [2 - (Q + 3\chi)] \frac{r_0}{T_H} = \left[\frac{2 - (Q + 3\chi)}{(Q - \chi)} \right] \pi r^2 \left(1 - \frac{4r_0}{r} \right)^{-(Q - \chi/2 - 1)} \Big|_{r=4r_0},$$

where we have used $M = M_t + M_s = 2r_0(Q - \chi)$. Note that the entropy S is inversely proportional to the temperature T_H and it does not satisfy the usual Bekenstein-Hawking relation [11], $S = (1/4) A = \pi r^2 (1 - 4r_0/r)^{-(Q-1)} |_{r=4r_0}$ as speculated earlier. Although we have obtained the Hawking temperature T_H , the local temperature $T(r)$ and the entropy S of our BD black hole in eq. (5), no definite statement concerning their behaviors can be made unless the precise estimation of the allowed values of the combination of parameters such as $(Q - \chi/2)$ and $(Q - \chi)$ is done. Therefore, next we carry out the estimation of allowed values of these parameters. In determining them, two bottom-line conditions are: i) the constraint equation in (4) that parameters Q and χ should satisfy must always be imposed and ii) the condition for the possible formation of non-trivial BD black holes given in eq. (7) must hold. First, as for the quantity $(Q - \chi/2)$, the condition ii) gives $(Q - \chi/2) > 1/2$ and $Q \geq 2$ whereas the condition i) leads to $-1 \leq (Q - \chi/2) \leq 1$ which, when put together, yields

$$(17) \quad \frac{1}{2} < \left(Q - \frac{\chi}{2} \right) \leq 1 \quad \text{and} \quad Q \geq 2.$$

Next, it is now the turn of the quantity $(Q - \chi)$ and the condition i) leads to

$$-\sqrt{\frac{2\omega + 4}{2\omega + 3}} \leq (Q - \chi) \leq \sqrt{\frac{2\omega + 4}{2\omega + 3}}$$

while the condition ii) or, equivalently, the condition obtained in eq. (17) gives in addition to $Q \geq 2$,

$$\left\{ (Q - \chi) \leq \frac{2\omega + 4}{2\omega + 3} \right\} \cap \left\{ (Q - \chi) < \frac{1}{2} \left(1 - \sqrt{\frac{3}{2\omega + 3}} \right) \text{ or } (Q - \chi) > \frac{1}{2} \left(1 + \sqrt{\frac{3}{2\omega + 3}} \right) \right\}$$

which, when put together, yield

$$(18) \quad -\sqrt{\frac{2\omega + 4}{2\omega + 3}} \leq (Q - \chi) < \frac{1}{2} \left(1 - \sqrt{\frac{3}{2\omega + 3}} \right) \quad \text{and} \quad Q \geq 2$$

or

$$\frac{1}{2} \left(1 + \sqrt{\frac{3}{2\omega + 3}} \right) < (Q - \chi) \leq \sqrt{\frac{2\omega + 4}{2\omega + 3}} \quad \text{and} \quad Q \geq 2.$$

Further, from the relationship between $\alpha \equiv (Q - \chi/2)$ and $\beta \equiv (Q - \chi)$, that is $\alpha = (2\omega + 4)^{-1} [(2\omega + 3)\beta \pm \sqrt{(2\omega + 4) - (2\omega + 3)\beta^2}]$, it is straightforward to see that,

for parameter values obtained in eqs. (17) and (18), the overlap $1/2 < \alpha \leq 1$ and

$$\frac{1}{2} \left(1 + \sqrt{\frac{3}{2\omega + 3}} \right) < \beta \leq \sqrt{\frac{2\omega + 4}{2\omega + 3}}$$

can happen as is manifest, for instance, for

$$\alpha = 1/\sqrt{2}, \quad \beta = \frac{1}{\sqrt{2}} \left(1 + \sqrt{\frac{1}{2\omega + 3}} \right)$$

whereas the other overlap can never happen since we assume $\omega > 0$. In addition, consider as cases of special interest, $Q = 1, \chi = 0$ and $Q = 1, \chi = 2/(\omega + 2)$. The first case is the Schwarzschild solution as mentioned earlier and the second case corresponds to a BD black hole with a singular horizon but satisfying the weak-energy condition on the BD scalar field. Finally, consider three cases (including two cases of special interest just discussed):

- case I $1/2 < (Q - \chi/2) < 1, \frac{1}{2} \left(1 + \sqrt{\frac{3}{2\omega + 3}} \right) < (Q - \chi) \leq \sqrt{\frac{2\omega + 4}{2\omega + 3}}$ and $Q \geq 2$;
- case II $Q = 1$ and $\chi = \frac{2}{\omega + 2}$;
- case III $Q = 1$ and $\chi = 0$.

The behaviors of thermodynamic functions $T_H, T(r)$ and S in eqs. (11), (13) and (16), respectively are given for each of the above three cases as

case I $T_H = \infty, T(r) = \infty, S = 0,$

case II $T_H = \infty, T(r) = \infty, S = 0,$

case III $T_H = \frac{1}{16\pi r_0}, T(r) = \frac{1}{16\pi r_0} \sqrt{\frac{r}{r - 4r_0}}, S = 16\pi r_0^2.$

This behavior of thermodynamic functions reveals that BD black-hole solutions with parameter values other than $Q = 1$ and $\chi = 0$ appear to be physically irrelevant and hence should be rejected as unphysical solutions. Obviously, this is based on the observation that for cases I and II, we have $S = 0$ and $T_H = \infty$ at *any stage* of the Hawking evaporation which is against our conventional wisdom based on the usual statistical mechanics. One might want to propose a different interpretation of these behaviors of thermodynamic quantities. Namely, since an infinite value of temperature might signal the breakdown of the semiclassical treatment involved in typical black-hole thermodynamics, it may well be that the non-trivial BD black holes could already be quantum entities which do not admit semiclassical treatment. It appears that the breakdown of the semiclassical approximations can be attributed either to the failure of weak-energy condition or to the singular behavior of the horizon. Indeed case I corresponds to regular black holes violating the weak-energy condition on Φ whereas case II represents a singular black hole satisfying the weak-energy condition. Generally, it seems that the non-singular behavior (which requires $Q \geq 2$) and the weak-energy condition on Φ (which demands $-1 \leq Q \leq 1$) cannot be simultaneously accommodated by any non-trivial BD black hole. However, since we can always retain non-singular behavior at the expense of the weak-energy condition (simply by

demanding $Q \geq 2$), the breakdown can be attributed solely to the latter. Thus to summarize, non-trivial BD black-hole solutions violating the weak-energy condition on Φ , that we allowed classically, turned out to be entities which demand more rigorous quantum treatment. Now we have two options; first, semiclassical analysis in the black-hole thermodynamics indeed works and hence the non-trivial BD black holes should be rejected as they fail this quantum censorship. Second, they are really quantum entities to which the conventional black-hole thermodynamics cannot be naively applied in the first place. In the absence of a consistent theory of quantum gravity, the second option is well beyond our scope and in this option we do not even know whether or not these non-trivial BD solutions can be identified with black holes at all. Thus in the present work, we choose to take a more conservative viewpoint, namely the first option.

To conclude, in the present work we examined the thermodynamic properties of all classically allowed static, spherically-symmetric black-hole solutions in vacuum BD theory. And the lesson we learned from our analysis is as follows: not all classically allowed black-hole solutions may be physically realistic. They should be further censored by the quantum nature of black holes, namely their Hawking evaporation mechanism. And when applied to the particular case discussed in the present work, it turned out that the non-trivial solutions that violate the weak-energy condition on the BD scalar field and thus have been discounted in Hawking's theorem fail to survive the quantum censorship and hence would not really arise in nature. After all, stationary black holes in BD theory appear to be identical to those in general relativity once they settle down.

REFERENCES

- [1] BRANS C. and DICKE R. H., *Phys. Rev.*, **124** (1961) 925.
- [2] See, for example, WILL C. M., *Theory and Experiment in Gravitational Physics*, revised edition (Cambridge University Press, Cambridge) 1993.
- [3] BRANS C., *Phys. Rev.*, **125** (1962) 2194.
- [4] CAMPANELLI M. and LOUSTO C. O., *Int. J. Mod. Phys. D*, **2** (1993) 451.
- [5] LEE D. L., *Phys. Rev. D*, **10** (1974) 2374.
- [6] S. W. HAWKING, *Commun. Math. Phys.*, **25** (1972) 167.
- [7] HAWKING S. W., *Commun. Math. Phys.*, **43** (1975) 199; HARTLE J. B. and HAWKING S. W., *Phys. Rev. D*, **13** (1976) 2188; WALD R. M., *Commun. Math. Phys.*, **45** (1975) 9.
- [8] GIBBONS G. W. and HAWKING S. W., *Phys. Rev. D*, **15** (1977) 2752.
- [9] WALD R. M., *General Relativity* (University of Chicago Press, Chicago) 1984.
- [10] TOLMAN R. C., *Relativity, Thermodynamics and Cosmology* (Clarendon Press, Oxford) 1931.
- [11] BEKENSTEIN J. D., *Phys. Rev. D*, **7** (1973) 2333; **9** (1974) 3292.