On the properties of Konishi-Kaneko map (*)

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Summary. — The existence of period-doubling bifurcations is shown for the Konishi-Kaneko map. The value of the control parameter k_∞ corresponding to the transition to the chaotic phase is estimated, along with the Feigenbaum universal number. This type of facts can be crucial in the diagnosis of chaos in stellar dynamics.

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An interesting possibility of study of the dynamics of N-body gravitating systems is provided by the Konishi-Kaneko iterated map [1-5]. Of particular interest is the agreement of the Konishi-Kaneko map with the thermodynamic considerations and the results of numerical simulation, as demonstrated by Inagaki [4].

At least two points outline the importance of iterated maps:

1) Iterated maps can enable one to avoid in a certain way the principal difficulties associated with N-body systems—the non-compactness of the phase space and singularity of Newtonian interaction.
2) Iterated maps can be rather informative in revealing the mechanisms of the development of chaos.

The Konishi-Kaneko map is defined as follows [1, 2]:

$$p_i^{n+1} = p_i^n + k \sum_{j=1}^{N} \sin 2\pi (x_j^n - x_i^n) , \tag{1}$$

$$x_i^{n+1} = x_i^n + p_i^{n+1} \quad (\text{mod } 1) . \tag{2}$$

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This system describes a 1-dimensional N-body system with a potential of interaction which is free of singularity and is attractive if $k > 0$. It represents a symplectic 2D map of interval (0, 1) and should have some common properties with maps known few decades ago, such as the Ulam map [6] proposed to describe the Fermi mechanism of acceleration of cosmic rays, and the map of Zaslavsky and Sinakh [7]. Among the recent interesting studies in this area we mention the papers by Kim (see, *e.g.*, [8]), where the symmetrically coupled two 1D systems are studied and the existence of Feigenbaum bifurcations is shown, with the value of the Feigenbaum constant $\delta = 8.72\ldots$. Chaotic properties of the present non-symmetrical, *i.e.* Konishi-Kaneko system were observed in [2], where the Lyapunov numbers of the system were calculated for clustered and non-clustered states. Note that at $N = 2$ we have an integrable system.

The Jacobian of the Konishi-Kaneko system is

$$\frac{\partial(y^{n+1}, x^{n+1})}{\partial(y^n, x^n)} = 1.$$

The corresponding Hamiltonian system was studied in the aspect of the thermodynamic instability.

Obviously, not every map defined on the (0, 1) interval can possess Feigenbaum bifurcations. Therefore first we have to check the necessary condition for the existence of period-doubling bifurcations, *i.e.* the negativity of the Schwartzian derivative:

$$Sf \equiv f'''/f'' - 3/2(f''/f')^2 < 0\,. \tag{3}$$

This condition is fulfilled for eqs. (1), (2) since $f'''/f' < 0$ for any value of k.

To obtain the bifurcation scale δ we have to find out the values of period-doubling bifurcation points which must satisfy the conditions

$$\sum_j^N |x_j^{n+1} - x_j^n| < \varepsilon, (2^1 = 2); \quad \sum_j^N |x_j^{n+2} - x_j^n| < \varepsilon, \sum_j^N |x_j^{n+3} - x_j^{n+1}| < \varepsilon, (2^2 = 4),$$

for each k_n, $n = 1, 2, \ldots$, respectively. ε is the accuracy of the obtained values of k_n. The accuracy of calculation of k_1, *e.g.*, for $N = 10$, was $\varepsilon \approx 10^{-5}$, and 10^{-4} for k_2 and k_3.

TABLE I.

$k_2 - k_1$	$k_3 - k_2$	δ
0.000196...	0.00002222...	8.82...
0.000194...	0.00002222...	8.78...
0.00019368...	0.00002222...	8.71584...
0.00019368...	0.00002210...	8.76359...
0.00019368...	0.00002225...	8.70490...
0.00019368...	0.00002223...	8.71219...
0.00019368...	0.00002221...	8.71950...
0.00019368...	0.00002219...	8.72682...
0.00019368...	0.000022205...	8.72315...
0.00019368...	0.000022209...	8.71950...
0.00019368...	0.000022207...	8.72315...

These calculations were enough to find out the Feigenbaum universal number $\delta = 8.72\ldots$. The results of calculations for $N = 10$ are given in table I.

The results, *e.g.*, for $N = 3, 5, 7$ were absolutely identical with those for $N = 10$, though with different accuracy ε. We have noticed a clear decrease in the accuracy with the increase of the number of particles.

The estimation of the values of k_n requires a careful calculation procedure because of the complicated character of the system and of the sensitivity on the iterations of k_n and on the accuracy ε.

Using the obtained values of k_n and the formula

$$k_\infty = (\delta k_{n+1} - k_n)/(\delta - 1)\,, \tag{4}$$

we also estimate the k_∞, from which the chaotic behavior of the system is established and the map never repeats itself:

$$k_\infty = 0.1307\ldots\,.$$

At $k > k_\infty$ the system should have positive Lyapunov numbers as shown in [2].

The period-doubling points correspond to the phase transitions of second order [9], and can enable the study of such systems via the methods of thermodynamic formalism [10].

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