The n-th–order virial theory. Permitted figures of equilibrium (*)

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Summary. — In the virial method we take moments of the equations of motion, from which we study the equilibrium and the stability of ellipsoidal figures. The virial equations of the various orders given by Chandrasekhar (*Ellipsoidal Figures of Equilibrium*, 2nd edition (Dover, New York, N.Y.) 1987) in the case of homogeneous and uniformly rotating masses, are generalized to nonuniformly rotating ellipsoidal configurations of nonhomogeneous density, with internal fluid motions of nonuniform vorticity. We derive the permitted equilibrium figures using the higher-order virial equations and we give a suitable classification.

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1. – Introduction

In the epilogue of his book *Ellipsoidal Figures of Equilibrium*, Chandrasekhar refers to a critic of Eddington to the study of homogeneous liquid masses: "It is not enough to deal with theoretical liquid masses. The astronomer wants to know how these results are maintained when we take into account the non-homogeneous or gaseous condition of actual stars." The virial method developed in Chandrasekhar [1], concerning exclusively homogeneous and uniformly rotating masses, consists in replacing the equation of motion governing a problem by its moments with respect to the coordinates; and these moment equations could furnish certain relationships that

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constrain the solutions in some way. In general, this method is not restricted to homogeneous masses, but it enables exact solutions only in that case.

The study of the gravitational equilibrium of heterogeneous rotating masses was considered by Hamy [2] and Dive [3] in an attempt to generalize Maclaurin and Jacobi homogeneous ellipsoids. Their results were, respectively: a heterogeneous mass stratified on ellipsoidal shells, uniformly rotating cannot exist; a rotating axisymmetric mass cannot be barotropic, it must be baroclinic (see also Tassoul [4]).

In this work, following and generalizing the previous results of Papers I to IX [5-13], we consider an inhomogeneous, rotating, self-gravitating fluid mass with anisotropic pressure and internal motions that are in an inertial frame multilinear functions of the coordinates. We propose n-th–order virial equations in a rotating frame and we discuss the constraints for the equilibrium of ellipsoidal configurations, coming from higher-order virial equations.

In sect. **2** the virial equations of n-th order are written in a very general form, using useful and compact definitions of the significative coefficients (for details see Paper IX). The most general velocity field into an ellipsoidal figure, preserving its form, as seen from a frame of reference where the ellipsoid is at rest (Paper VIII), producing internal fluid motions of nonuniform vorticity, is considered in sect. **3**. Here the set of virial equations is specialized to the case of S-type Riemann ellipsoids. Because only the even-order virial equations are different from zero, further constraints for the equilibrium coming from higher-order virial equations are discussed. In sect. **4** we consider the Dedekind theorem and the generalized Jacobi, Dedekind and Riemann ellipsoids. Conclusions are drawn in sect. **5**.

2. – The virial equations of n-th order

We consider an ideal self-gravitating fluid of density $\varrho(\mathbf{x}, t)$ and anisotropic pressure P, rotating with an angular velocity $\mathbf{\Omega}$ (Chandrasekhar [1]). In a rotating frame, the virial equations of order n may be generated by multiplying the hydrodynamic equation by $x_i^{a-1} x_j^b x_k^c$ and integrating over the volume V. The index i appears $a-1$ times, the indices j, k appear b and c times, respectively; and $i \neq j \neq k$; $a \geq 1$, $b \geq 0$, $c \geq 0$; $a+b+c=n$. Thus

$$\int \varrho \frac{\mathrm{d}u_i}{\mathrm{d}t} x_i^{a-1} x_j^b x_k^c \,\mathrm{d}V = -\sum_{lmpq} \varepsilon_{ilm} \varepsilon_{mpq} \Omega_l \Omega_p \int \varrho x_i^{a-1} x_j^b x_k^c \,\mathrm{d}V + 2\sum_{lm} \varepsilon_{ilm} \Omega_m \int \varrho u_l x_i^{a-1} x_j^b x_k^c \,\mathrm{d}V - \sum_{lm} \varepsilon_{ilm} \dot{\Omega}_l \int \varrho x_i^{a-1} x_j^b x_k^c \,\mathrm{d}V - \sum_l \int \partial_l P_{li} x_i^{a-1} x_j^b x_k^c \,\mathrm{d}V + \int \varrho \partial_i \mathcal{V} x_i^{a-1} x_j^b x_k^c \,\mathrm{d}V . \tag{1}$$

The gravitational potential $\mathcal{V}$ satisfies the Poisson equation $\nabla^2 \mathcal{V} = -4\pi G \varrho$ and the boundary of the configuration is defined by $P_{li} = 0$.

The generalized moments are: the n-th moment of inertia tensor: $I_{i,j,k}^{a,b,c} = \int \varrho x_i^a x_j^b x_k^c \,\mathrm{d}V$, the n-th moment of the kinetic-energy tensor: $2T_{i,j,k}^{a,b,c} = \int \varrho u_i u_j x_i^a x_j^b x_k^c \,\mathrm{d}V$, the n-th moment of the potential energy tensor: $W_{i,j,k}^{a,b,c} = \int \varrho \partial_i \mathcal{V} x_i^{a-1} x_j^b x_k^c \,\mathrm{d}V$, and finally the $(n-2)$-th moment of the pressure tensor:

$\Pi^{a-2,\,b,\,c}_{i,\,j,\,k} = \int P_i x_i^{a-2} x_j^b x_k^c \,\mathrm{d}V$. Following the previous definitions eq. (1) may be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \varrho u_i x_i^{a-1} x_j^b x_k^c \,\mathrm{d}V = 2[(a-1)\,T_{ii,\,a-2,\,b,\,c} + bT_{ij,\,a-1,\,b-1,\,c} + cT_{ik,\,a-1,\,b,\,c-1}] + \tag{2}$$

$$+ W_{i,\,a-1,\,b,\,c} + \Omega^2 I_{a,\,b,\,c} - \Omega_i \sum_{p=1}^{3} \Omega_p I_{p,\,a-1,\,b,\,c} +$$

$$+ 2\sum_{lm} \varepsilon_{ilm} \Omega_m \int \varrho u_l x_i^{a-1} x_j^b x_k^c \,\mathrm{d}V - \sum_{lm} \varepsilon_{ilm} \dot{\Omega}_l I_{a-1,\,b,\,c} + (a-1)\,\Pi_{ii,\,a-2,\,b,\,c}.$$

Under conditions of a stationary state, the left-hand–side term and $\dot{\Omega}_l$ are equal to zero.

3. – The n-th–order virial equations for heterogeneous ellipsoids

We now turn our attention to a class of heterogeneous ellipsoids, in which the strata of equal density are similar to and concentric with the bounding ellipsoid $\left(\sum_{i=1}^{3} \frac{x_i^2}{a_i^2} = m^2,\ \varrho = \varrho_{\mathrm{c}} f(m^2)\right)$, a_i (semi-axes of the bounding ellipsoid). The Newtonian potential has the form (Chandrasekhar [1], sect. 20)

$$\mathcal{V} = \pi G a_1 a_2 a_3 \int_0^\infty \frac{\mathrm{d}u}{\Delta} F(m^2(u)) = (\pi G \varrho_{\mathrm{c}})\, a_1 a_2 a_3 \int_0^\infty \frac{\mathrm{d}u}{\Delta} \tilde{F}(m^2(u)) \tag{3}$$

with $F(m^2(u)) = \int_{m^2(u)}^1 \varrho(m^2)\,\mathrm{d}m = \varrho_{\mathrm{c}} \tilde{F}(m^2(u))$ and $m^2(u) = \sum_{i=1}^{3} \frac{x_i^2}{a_i^2 + u}$; thus the space derivative of $\mathcal{V}$ is

$$\partial_i \mathcal{V} = +\,2(\pi G \varrho_{\mathrm{c}})\, a_1 a_2 a_3 x_i \int_0^\infty \frac{\mathrm{d}u}{\Delta(a_i^2+u)} \frac{\mathrm{d}\tilde{F}(m^2(u))}{\mathrm{d}m^2(u)} = -2(\pi G \varrho_{\mathrm{c}})\, x_i C_i(\mathbf{x}), \tag{4}$$

where $C_i(\mathbf{x})$ are the elements of a diagonal matrix, explicitly

$$C_i(\mathbf{x}) = -a_1 a_2 a_3 \int_0^\infty \frac{\mathrm{d}u}{\Delta(a_i^2+u)} \frac{\mathrm{d}\tilde{F}(m^2(u))}{\mathrm{d}m^2(u)} = a_1 a_2 a_3 \int_0^\infty \frac{\mathrm{d}u}{\Delta(a_i^2+u)} f(m^2(u))\,. \tag{5}$$

C_i reduces to the index symbol defined in Chandrasekhar [1], sect. 21, in the homogeneous case. By definition $\Delta^2 = (a_1^2+u)(a_2^2+u)(a_3^2+u)$, and $\varrho(m^2) = \varrho_{\mathrm{c}} f(m^2)$.

For the velocity field describing the internal motions, we consider the general form (Paper VIII)

$$u_i = -\tilde{\phi} \sum_{j=1}^{3} (AZA^{-1})_{ij} x_j, \tag{6}$$

where the explicit form of $\tilde{\phi}$ will be specified in the following. A is a diagonal matrix having the values of semi-axes a_i as elements, and $Z_{ij} = \frac{N_{ij}}{a_i a_j} C = \sum_k \varepsilon_{ijk} Z_k$. Here Z_{ij} is defined in such a way that the factor $1/C$ gives to $\mathbf{Z}$ the same dimension of $\boldsymbol{\Omega}$. N_{ij} is the dual of an axial vector $\hat{\mathbf{n}}$, which determines the direction of $\boldsymbol{\Omega}$ and $\mathbf{Z}$.

Considering the steady state, it follows from eq. (2) and the relation $(AZA^{-1})_{ij} = S_{ij}$ that

$$(7)\qquad \sum_{pq=1}^{3}[(a-1)\,S_{ip}S_{iq}K_{pq,\,i,\,j,\,k}^{a-2,\,b,\,c} + bS_{ip}S_{jq}K_{pq,\,i,\,j,\,k}^{a-1,\,b-1,\,c} + cS_{ip}S_{kq}K_{pq,\,i,\,j,\,k}^{a-1,\,b,\,c-1}] +$$

$$+W_{i,\,j,\,k}^{a,\,b,\,c} + \Omega^2 I_{i,\,j,\,k}^{a,\,b,\,c} - \Omega_i \sum_{p=1}^{3}\Omega_p I_{p,\,i,\,j,\,k}^{a-1,\,b,\,c} - 2\sqrt{\frac{\gamma}{\alpha}}\sum_{lmp}\varepsilon_{ilm}\,\Omega_m S_{lp} J_{p,\,i,\,j,\,k}^{a-1,\,b,\,c} =$$

$$= -(a-1)\,\Pi^{a-2,\,b,\,c}_{i,\,j,\,k},$$

where $K_{pq,\,i,\,j,\,k}^{a-1,\,b-1,\,c} = \int \varrho\,\tilde{\phi}^2 x_p x_q x_i^{a-2} x_j^b x_k^c \,\mathrm{d}V$, $I_{p,\,i,\,j,\,k}^{a-1,\,b,\,c} = \int \varrho x_p x_i^{a-1} x_j^b x_k^c \,\mathrm{d}V$, $J_{p,\,i,\,j,\,k}^{a-1,\,b,\,c} = \int \varrho\,\tilde{\phi}\, x_p x_i^{a-1} x_j^b x_k^c \,\mathrm{d}V$, and p, q take values from the set $\{i, j, k\}$ with $i \neq j \neq k$.

It can be shown that these virial equations with odd values of $n = a + b + c$ are identically zero (Paper IX). The analysis may be performed easily by classifying the powers of the coordinates. In fact there are only the following possibilities: 1) $a =$ even; $b, c =$ odd; 2) $b =$ even; $a = c =$ odd; 3) $c =$ even; $a, b =$ odd; 4) $a, b, c =$ even. The cases 2) and 3) are equivalent due to the interchangeable positions of j and k in eq. (7). The cases 1) and 2) are non-equivalent, due to the privileged position of the index i. From now on, we restrict the analysis to S-type Riemann ellipsoids, where $\hat{n} = (0, 0, 1)$, $\mathbf{\Omega} = (0, 0, \Omega)$, $\mathbf{Z} = (0, 0, Z)$. For this condition only the virial equations of case 4) are different from zero.

Case 4) $a, b, c =$ even. Equation (7) becomes

$$(8)\qquad Z_j^2\left((a-1)\,\frac{a_i^2}{a_k^2}\,K_{i,\,j,\,k}^{a-2,\,b,\,c+2} - cK_{i,\,j,\,k}^{a,\,b,\,c}\right) + Z_k^2\left((a-1)\,\frac{a_i^2}{a_j^2}\,K_{i,\,j,\,k}^{a-2,\,b+2,\,c} - bK_{i,\,j,\,k}^{a,\,b,\,c}\right) +$$

$$+(\Omega_j^2 + \Omega_k^2)\,I_{i,\,j,\,k}^{a,\,b,\,c} + 2\left(\Omega_j Z_j\,\frac{a_k}{a_i} + \Omega_k Z_k\,\frac{a_j}{a_i}\right) J_{i,\,j,\,k}^{a,\,b,\,c} + W_{i,\,j,\,k}^{a,\,b,\,c} + (a-1)\,\Pi_{i,\,j,\,k}^{a-2,\,b,\,c} = 0\,.$$

Using the reduced (dimensionless) cylindrical coordinates $\tilde{x}_1 = \tilde{r}\cos\theta$, $\tilde{x}_2 = \tilde{r}\sin\theta$, and $\tilde{\phi} = \tilde{\phi}(m^2, \tilde{r}^2, \tilde{x}_3^2)$, it is easy to show that

$$\frac{(a-1)\,\dfrac{a_i^2}{a_k^2}\,K_{i,\,j,\,k}^{a-2,\,b,\,c+2}}{K_{i,\,j,\,k}^{a,\,b,\,c}} = c+1 \quad\text{and}\quad \frac{(a-1)\,\dfrac{a_i^2}{a_j^2}\,K_{i,\,j,\,k}^{a-2,\,b+2,\,c}}{K_{i,\,j,\,k}^{a,\,b,\,c}} = b+1\,,$$

so that eq. (8) takes the explicit form

$$(9)\qquad Z^2 K_{1,\,2,\,3}^{a,\,b,\,c} + \Omega^2 I_{1,\,2,\,3}^{a,\,b,\,c} + 2\Omega Z\,\frac{a_2}{a_1} + J_{1,\,2,\,3}^{a,\,b,\,c} + W_{1,\,2,\,3}^{a,\,b,\,c} + (a-1)\,\Pi_{1,\,2,\,3}^{a-2,\,b,\,c} = 0\,,$$

$$(10)\qquad Z^2 K_{2,\,1,\,3}^{a,\,b,\,c} + \Omega^2 I_{2,\,1,\,3}^{a,\,b,\,c} + 2\Omega Z\,\frac{a_1}{a_2} + J_{2,\,1,\,3}^{a,\,b,\,c} + W_{2,\,1,\,3}^{a,\,b,\,c} + (a-1)\Pi_{2,\,1,\,3}^{a-2,\,b,\,c} = 0\,,$$

$$(11)\qquad W_{3,\,1,\,2}^{a,\,b,\,c} + (a-1)\,\Pi_{3,\,1,\,2}^{a-2,\,b,\,c} = 0\,.$$

It is easy to show that in the linear and homogeneous case ($\tilde{\phi} = f = 1$, $P_i = P_{ic}(1 - m^2)$), this set reduces to just three equations and it coincides with the hydrodynamic equations (Chandrasekhar [1], Paper VIII).

4. – Dedekind's theorem. Heterogeneous Jacobi, Dedekind and Riemann ellipsoids

Dedekind's theorem, and the two adjoint sequences of Jacobi ("pure" rotating systems) and Dedekind ellipsoids (non-rotating systems, maintainig their figure by internal streaming only), can be generalized to the case of heterogeneous density. From eqs. (9)-(11) we can discuss an important class of equilibrium solutions: the self-adjoint (Dedekind's theorem): $\Omega^\dagger = Z$, $Z^\dagger = \Omega$; († indicates Dedekind conjugation: matrix transposition and exchange of Ω and Z). This operation does not change the form of equations so that they are self-adjoint just in the linear case ($\tilde{\phi} = 1$).

As a consequence: *Dedekind's theorem about the existence of a Dedekind ellipsoid, corresponding to a Jacobi ellipsoid or, in general, the existence of self-adjoint configurations, is limited to the linear case. The density may be inhomogeneous and the pressure may be anisotropic.* This result is also given in Chambat [14].

4·1. *Generalized Maclaurin spheroids and Jacobi ellipsoids.* – We further specialise our analysis to the homogeneous and heterogeneous figures of S-type, spherically, spheroidally and ellipsoidally stratified, in the case of uniform rotation ($\mathbf{\Omega} = \text{const}$, $\mathbf{Z} = 0$). We include here as particular cases the classical Maclaurin spheroids and Jacobi ellipsoids [1]. We generalize these equilibrium sequences to heterogeneous density and anisotropic pressure.

From eqs. (9) to (11) for rotating ellipsoids—we will consider the second- and fourth-order virial equations—the higher orders give the same information. We can perform the analysis, beginning with the *sphere*. In this case $a_1 = a_2 = a_3$. The conclusion is that *the uniformly rotating sphere, homogeneous or heterogeneous, must be anisotropic and the pressure must be spherically stratified.*

Case of *spheroids*: *a*) In the case of homogeneous oblate spheroids ($a_1 > a_3$), we have that *homogeneous, oblate Maclaurin spheroids exist, can be isotropic* ($P_1 = P_2 = P_3$), *or anisotropic* ($P_1 = P_2 \neq P_3$), *with the pressure spheroidally stratified.* This last result generalizes the solution of Chandrasekhar [1], chapt. 5.

b) In the case of homogeneous prolate spheroids ($a_3 > a_1$), the anisotropy must give $P_1 < P_3$, then: *homogeneous, prolate Maclaurin spheroids exist and must be anisotropic* ($P_3 > P_1$) *and the pressure spheroidally stratified.*

We consider, finally, the heterogeneous case. Thus *heterogeneous, isotropic or anisotropic, barotropic Maclaurin spheroids cannot exist.* This conclusion generalizes Dive's results [3] to the anisotropic case.

Instead: *heterogeneous, baroclinic Maclaurin spheroids are allowed.*

Case of *ellipsoids*: we conclude the non-existence of heterogeneous, barotropic ellipsoids. They exist in the homogeneous case with ellipsoidal stratifications of the isotropic pressure (the Jacobi ellipsoids in Chandrasekhar [1] sect. 39). In addition, there exist homogeneous ellipsoids having the pressure ellipsoidally stratified and anisotropic ($P_1 \neq P_2 \neq P_3$).

4·2. *Generalized Dedekind ellipsoids.* – We study now the heterogeneous figures of S-type, spherically, spheroidally and ellipsoidally stratified, in the case of non-uniform vorticity ($Z \neq 0$) (differential rotation), without figure rotation ($\mathbf{\Omega} = 0$). As in the previous case, we will consider only the 2nd- and 4th-order virial equations. The higher-order virial equations are identically satisfied and do not give any new piece of information.

In the following we consider two independent cases in which the form of velocity field is, respectively, $\tilde{\phi} = \tilde{\phi}(m^2)$ and $\tilde{\phi} = \tilde{\phi}^*$ indicating the general forms: $\tilde{\phi}(m^2, \tilde{r}^2, \tilde{x}_3^2)$, $\tilde{\phi}(m^2, \tilde{r}^2)$, $\tilde{\phi}(m^2, \tilde{x}_3^2)$, $\tilde{\phi}(\tilde{r}^2, \tilde{x}_3^2)$, $\tilde{\phi}(\tilde{x}_3^2)$, $\tilde{\phi}(\tilde{r}^2)$, with $m^2 = \tilde{r}^2 + \tilde{x}_3^2$, $\tilde{r}^2 = \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2}$.

a) For $\tilde{\phi} = \tilde{\phi}(m^2)$.

Summarizing:

1) *A sphere having non-uniform vorticity (differential rotation), homogeneous or heterogeneous, must be anisotropic with* $P_3 > P_1$.
2) *There exists a homogeneous or heterogeneous sphere, having* $\tilde{\phi} = \tilde{\phi}(\tilde{r}^2)$ *and the pressure radially stratified,* $P_1(r) \neq P_3(r)$.
3) *There exist homogeneous spheroids having* $\tilde{\phi} = \tilde{\phi}(m^2)$*: the oblate ones can be isotropic or anisotropic; the prolate ones must be anisotropic with* $P_1 < P_3$*, the pressure being spheroidally stratified.*
4) *The heterogeneous spheroids must be baroclinic* (*see* [3]).
5) *The homogeneous Dedekind's ellipsoids with pressure ellipsoidally stratified isotropic or anisotropic are classifiable as oblate, isotropic and anisotropic, or prolate and anisotropic with* $P_1 < P_3$, $P_2 < P_3$. These results generalize the oblate, isotropic configurations given in Chandrasekhar [1], sect. 39.
6) *Generalized heterogeneous Dedekind ellipsoids cannot exist.* This conclusion agrees with the recent results obtained by Chambat [14], using the hydrodynamic equations.

b) We now assume the other forms of $\tilde{\phi}$, denoted as $\tilde{\phi}^*$.

We conclude that *there exist homogeneous and heterogeneous, anisotropic spheres with non-uniform vorticity (differential rotation) of the form* $\tilde{\phi}^*$.

We consider now the spheroidal case:

the homogeneous spheroids with non-uniform vorticity (differential rotation) of the form $\tilde{\phi}^*$ *exist and their pressure is spheroidally stratified*;

the heterogeneous spheroids with differential rotation of the form $\tilde{\phi}^*$ *exist and must be baroclinic*. These results generalize the conclusions obtained by Dive [3].

In the case of the ellipsoids: *the heterogeneous ellipsoids with* $\tilde{\phi} = \tilde{\phi}^*$ *do not exist.*

4·3. *Generalized Riemann ellipsoids.* – We now turn to the case of the generalized *S*-type Riemann ellipsoids, with uniform figure rotation ($\mathbf{\Omega} \neq 0$) and non-uniform vorticity ($\mathbf{Z} \neq 0$): The analysis performed on the 2nd- and 4th-order virial equations gives: *there do not exist heterogeneous generalized Riemann ellipsoids with* $\tilde{\phi} = \tilde{\phi}(m^2)$. This is a generalization of the result obtained by Chambat [14] in the linear case ($\tilde{\phi} = 1$).

For spheres and spheroids, we have the result: *a homogeneous or heterogeneous sphere must be anisotropic if* $\mathbf{\Omega}$ *and* $\mathbf{Z}$ *are parallel, with* P_1, P_3 *spherically stratified. If* $\mathbf{\Omega}$ *and* $\mathbf{Z}$ *are antiparallel the sphere can be isotropic.*

A homogeneous spheroid with $\tilde{\phi} = \tilde{\phi}(m^2)$ *must be barotropic. If* $\mathbf{\Omega}$ *and* $\mathbf{Z}$ *are parallel, then one must have* $P_1 < P_3$. *If* $\mathbf{\Omega}$ *and* $\mathbf{Z}$ *are antiparallel the spheroid can be isotropic or anisotropic.*

A heterogeneous spheroid must be baroclinic and can be isotropic or anisotropic.

In the case $\tilde{\phi} = \tilde{\phi}^*$, *homogeneous and heterogeneous spheres and spheroids exist, with* $\mathbf{\Omega}$ *and* $\mathbf{Z}$ *parallel or antiparallel.*

As in the previous case, the spheroids are classifiable as oblate and prolate and are barotropic. Finally, let us consider the case of ellipsoids: *the homogeneous, isotropic or anisotropic Riemann ellipsoids exist and can be barotropic and baroclinic in* P_1, P_2, *but* P_3 *must be barotropic. The heterogeneous, generalized Riemann ellipsoids having* $\tilde{\phi}^*$ *cannot exist.*

5. – Conclusions

The tensor virial equations for a self-gravitating, rotating fluid mass are generalized to the *n*-th order using useful and compact definitions of the significative coefficients. The necessary equilibrium conditions coming from 2nd- and 4th-order virial equations are obtained for the case of *S*-type Riemann ellipsoids. The higher-order virial equations do not produce new information with respect to what established from just the 2nd- and 4th-order virial equations.

1) About the generalized *Dedekind theorem*: it is valid only in the case of linear velocity field (the density could be heterogeneous and the pressure anisotropic). This result is also given in Chambat [14].

2) *Jacobi configurations*: we consider the case of uniform rotation. Homogeneous and heterogeneous *spheres* exist, it must be anisotropic and barotropic (the pressure is spherically stratified as the bounding surface), and $P_1 < P_3$. Homogeneous Jacobi *spheroids* can be oblate and barotropic, the pressure can be isotropic or anisotropic. Prolate, barotropic configurations are permitted only if the pressure is anisotropic ($P_1 < P_3$). The heterogeneous Jacobi spheroids must be baroclinic (the pressure is a function of the coordinates as $P = P(\tilde{x}_1^2 + \tilde{x}_2^2, \tilde{x}_3^2)$), can be isotropic or anisotropic), this result was originally established by Dive [3] (see, *e.g.*, Tassoul [4], chapt. 4). The homogeneous barotropic Jacobi *ellipsoids* exist and can be oblate, isotropic or anisotropic, and prolate anisotropic ($P_1 < P_3$, $P_2 < P_3$). The heterogeneous Jacobi ellipsoids cannot exist according to the Hamy theorem [2].

3) *Dedekind configurations*: In this case the figures have only differential rotation with $\mathbf{\Omega} = 0$.

The homogeneous *spheres* must be barotropic and anisotropic ($P_1 < P_3$) in the case of velocity field $\tilde{\phi}$ and with $\tilde{\phi}^*$ can be isotropic and barotropic, or anisotropic ($P_1 < P_3$) and P_1 barotropic or baroclinic, P_3 barotropic. The heterogeneous spheres in both cases of $\tilde{\phi}$ and $\tilde{\phi}^*$, must be barotropic and anisotropic ($P_1 < P_3$).

The homogeneous Dedekind *spheroids* with $\tilde{\phi}$ can be oblate and barotropic, the pressure can be isotropic or anisotropic. Prolate, barotropic configurations are permitted only if the pressure is anisotropic ($P_1 < P_3$). In the case of $\tilde{\phi}^*$ the homogeneous spheroids can be oblate, barotropic, isotropic or anisotropic (P_1

barotropic or baroclinic, P_3 barotropic). The prolate spheroids must be anisotropic $P_1 < P_3$ and P_1 barotropic or baroclinic, P_3 barotropic. The heterogeneous Dedekind spheroids with $\tilde{\phi}$ must be baroclinic, can be isotropic or anisotropic; this result was originally established by Dive [3]. In the case of $\tilde{\phi}^*$ the heterogeneous spheroids can be isotropic and baroclinic or anisotropic (P_1 barotropic or baroclinic, P_3 baroclinic).

The homogeneous Dedekind *ellipsoids* with $\tilde{\phi}$ and $\tilde{\phi}^*$ can be oblate and barotropic, the pressure can be isotropic or anisotropic. Prolate, barotropic configurations are permitted only if the pressure is anisotropic ($P_1 < P_3$).

The heterogeneous Dedekind ellipsoids cannot exist according to Chambat [14].

4) *Riemann configurations*: In this case the figures have differential rotation and $\mathbf{\Omega} \neq 0$. The velocity fields considered are as before: $\tilde{\phi}$ and $\tilde{\phi}^*$.

The homogeneous and heterogeneous *spheres* must be barotropic in the cases of velocity field $\tilde{\phi}$ and $\tilde{\phi}^*$; they can be isotropic or anisotropic if $\mathbf{\Omega}$ and $\mathbf{Z}$ are antiparallel, or anisotropic ($P_1 < P_3$) if $\mathbf{\Omega}$ and $\mathbf{Z}$ are parallel.

The homogeneous Riemann *spheroids* with $\tilde{\phi}$ and $\tilde{\phi}^*$ can be oblate and barotropic, the pressure can be isotropic or anisotropic if $\mathbf{\Omega}$ and $\mathbf{Z}$ are antiparallel. Prolate, barotropic configurations are permitted only if the pressure is anisotropic ($P_1 < P_3$), with $\mathbf{\Omega}$ and $\mathbf{Z}$ parallel.

The heterogeneous Riemann spheroids with $\tilde{\phi}$ must be baroclinic, oblate or prolate, the pressure can be isotropic or anisotropic with $\mathbf{\Omega}$ and $\mathbf{Z}$ parallel or antiparallel. The heterogeneous Riemann spheroids with $\tilde{\phi}^*$ can be oblate, isotropic or anisotropic and baroclinic, with Ω and Z parallel; or prolate with P_1 barotropic or baroclinic and P_3 baroclinic, with Ω and Z antiparallel.

The homogeneous Riemann *ellipsoids* with $\tilde{\phi}$ can be oblate and barotropic, the pressure can be isotropic or anisotropic if $\mathbf{\Omega}$ and $\mathbf{Z}$ are antiparallel. Prolate, barotropic configurations are permitted only if the pressure is anisotropic ($P_1 < P_3$, $P_2 < P_3$), with $\mathbf{\Omega}$ and $\mathbf{Z}$ parallel. In the case of spheroids with $\tilde{\phi}^*$, it can be isotropic and barotropic or anisotropic and P_1 barotropic or baroclinic and P_3 barotropic.

The heterogeneous Riemann ellipsoids cannot exist.

In this work we recovered some classical results obtained by Dive [3] (a stratified heterogeneous spheroid, rotating and without differential rotation cannot be a barotrope) and generalized to ellipsoidal, anisotropic configurations. The Hamy theorem (a mass ellipsoidally stratified cannot have a uniform rotation) is confirmed also for the anisotropic case. All the Chandrasekar results for homogeneous configurations (see Chandrasekhar [1]), are generalized to the anisotropic case.

The virial equations of n-th order prove the non-existence of triaxial, stratified, heterogeneous equilibrium ellipsoids. Only a certain class of axisymmetric equilibrium figures with differential rotation (having or not) rigid rotation can exist (condition only necessary but not sufficient).

Tensor virial equations of higher orders can be used for the construction of the equilibrium configurations for the description of stellar systems or galaxies in oblate or prolate cases. Methods recently formulated by various authors require to satisfy the conditions of mechanical equilibrium in the average sense of the virial equations.

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