On metastable Einstein's clusters (*)

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Summary. — Specific and explicit examples of Einstein's clusters are given. We introduce the concept of metastable clusters and we analyze, as well, the special cases of clusters endowed with an infinite central gravitational redshift.

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1. – Introduction

The problem of general-relativistic star clusters was first introduced by Einstein in a classical paper in 1939 [1].

He proposed a model for spherical relativistic clusters that is conceptually very simple; its basic hypothesis is that all the particles constituting the system, self-gravitating, move on circular paths around the center of symmetry of the cluster.

The spherical symmetry is guaranteed by the hypothesis that the normals to the orbital planes of the particles constituting every shell of the cluster are distributed at random in all directions and also the phase angles of the paths are subject to a random distribution.

In the Einstein model, all the particles being centrifugally supported, it is possible to choose arbitrarily the number density of the particles $n(r)$ or, equivalently, the energy density of the cluster.

Einstein's results have been often reexamined, *e.g.*, by Zapolsky [2], Gilbert [3], Hogan [4] and Florides [5]. Special attention was given to the analysis of the stability of Einstein's clusters.

A different approach to the problem of relativistic clusters was proposed, still in the case of spherical symmetry and General Relativity, by Zel′dovich and Podurets [6]. Contrary to the work of Einstein, where no hypothesis was made on the equation of

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state of the system, they assumed a classical phase space distribution consisting in a suitably modified Maxwellian distribution:

$$(1)\qquad \begin{cases} f = Be^{-E/T} & \text{for } E \leqslant E_{\text{cut}} \equiv mc^2 - T/2\,, \\ f = 0 & \text{for } E > E_{\text{cut}}\,, \end{cases}$$

where E is the total energy of the "particle" and T is the temperature "measured by an infinetely remote observer", in energy units, constant on each single configuration. Using this truncated distribution, Zel′dovich and Podurets obtained the onset of gravitational instability at a central gravitational redshift $z \geqslant 1.5$.

This work was further generalized to the distribution function

$$(2)\qquad \begin{cases} f = Be^{-E/T} & \text{for } E \leqslant E_{\text{cut}} \equiv mc^2 - \alpha T/2\,, \\ f = 0 & \text{for } E > E_{\text{cut}} \end{cases}$$

(where α is a parameter constant over the single configuration) by Ipser (1969, 1980) [7-9], Fackerell (1970) [10], Shapiro and Teukolsky (1985) [11-13], Rasio *et al.* (1988, 1989) [14, 15], Merafina and Ruffini (1995) [16], Bisnovatyi-Kogan *et al.* (1993, 1995) [17, 18]. The main new result of these papers by comparison with the Zel′dovich and Podurets work is that stable relativistic clusters can exist all the way up to an infinite central gravitational redshift.

These studies have become of great interest as a consequence of the observations of active galactic nuclei.

Recently, the analysis of stimulated MASER emission around some of these galactic nuclei has permitted to obtain detailed information about their gravitational field, thanks to the necessary condition to be fulfilled in order to have stimulated emission (see, *e.g.*, Ruffini and Stella [19]).

The main issue still unsolved is how and when these highly relativistic regimes in the central cores have developed.

Our aim is to explore if stable clusters with different distribution functions than the ones given by eq. (2) can exist.

As a first step, we go back to the work of Einstein to exploit the freedom of $n(r)$ to analyze stable or metastable configurations, generalizing the choices initially made by Einstein, postponing the issue of the distribution function generating such structures to a further analysis.

In order to better understand the physical conditions occurring in centrifugally supported orbits in General Relativity within a cluster, we first considered the simpler problem of circular orbits in the Interior Schwarzschild Metric ([20], hereafter referred to as paper I); the results of that work are briefly summarized in sect. **2**.

In sect. **3** of this paper II, exploiting the results of paper I, we reformulate the Einstein model in ordinary (non-isotropic) spherical coordinates and we analyze the three following special cases:

a) $\tilde{\varrho}(r) = \text{const}$ (constant energy density),

b) $n(r) = \text{const}$,

c) $n(r) = n(0)[1 - \exp[1 - R/r]]$

(where R is the radius of the configuration) with particular attention to the behaviour of the effective potential, in order to study the stability of circular orbits.

In sect. **4** we introduce a general criterion of stability for Einstein's clusters, on the basis of the analysis of the stability of circular orbits, and we point out a novel result on the existence of metastable configurations, namely locally bound by the gravitational field but globally unbound.

In sect. **5** we estimate the limits of the central gravitational redshifts in the above configurations and conclude that, indeed, infinite redshift at the center of the configuration can be obtained in a variety of cases.

Finally, in the conclusions, we point out some new direction of ingoing work, relaxing the condition of spherical symmetry in favour of axially symmetric configurations, following the works of Morgan and Morgan [14] and Bardeen and Wagoner [21, 22].

2. – Circular orbits in the Interior Schwarzschild Metric

In this section we briefly recall the results of paper I, analyzing circular orbits in the Interior Schwarzschild Metric, namely the metric describing the gravitational field in the interior of a static and spherically symmetric fluid with constant energy density.

As usual, this metric can be expressed in the form

$$\mathrm{d}s^2 = b(r)\,\mathrm{d}t^2 - a(r)\,\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) \tag{3}$$

with

$$a(r) = \left(1 - \frac{2M}{R^3} r^2\right)^{-1},$$

$$b(r) = \left[\frac{3}{2}\left(1 - \frac{2M}{R}\right)^{1/2} - \frac{1}{2}\left(1 - \frac{2M}{R^3} r^2\right)^{1/2}\right]^2$$

(here and in the following we use geometrical units with $G = c = 1$; M and R are, respectively, the total mass and the radius of the fluid sphere).

To study the orbits of a test particle moving in this metric, we have to consider the geodesic equations :

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s} = 0\,.$$

It then follows that the motion of the particle is governed by the radial effective potential:

$$U_{\mathrm{eff}} = \sqrt{b(r)\left(m^2 + \frac{\Phi^2}{r^2}\right)} \tag{4}$$

(where Φ and m are, respectively, the angular momentum and the mass of the test particle).

Obviously, the circular orbits correspond to the extremals of the effective potential and they are, respectevely, stable or unstable if they correspond to a minimum or to a maximum.

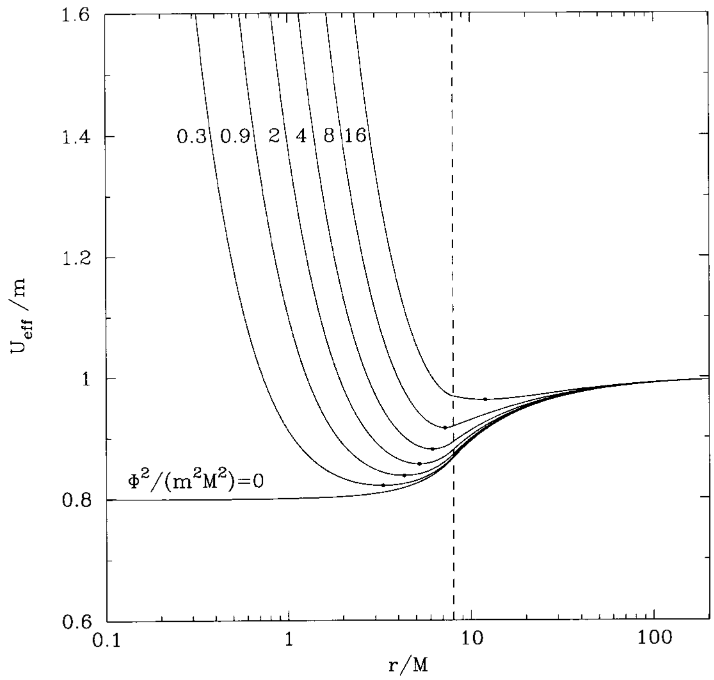

Fig. 1. – U_{eff}/m in the case of a fluid sphere with constant energy density and $R = 8M$, for selected values of $\Phi^2/(m^2M^2)$. As Φ^2 increases, the minimum moves from the interior to the exterior of the configuration. Dots show circular orbits.

The results are markedly different from the ones of the Exterior Schwarzschild Solution summarized, *e.g.*, in Landau L. D. and Lifšits E. M. [23].

The behaviour of the effective potential changes with the radius of the configuration and with the angular momentum of the test particle.

We can distinguish three cases:

1) $R > 6M$,

2) $3M < R < 6M$,

3) $(9/4)\, M < R < 3M$.

In the first case (fig. 1) the effective potential has always only one minimum; as the value of the angular-momentum increases, the circular orbits move from the interior to the exterior of the configuration and the corresponding energy also increases. (Clearly in the exterior of the configuration the metric we consider is the Exterior Schwarzschild Metric.) This means that, in this case, stable circular orbits exist for all the possible values of r and also for $0 \leqslant \Phi < \infty$.

In the second case (fig. 2), for low values of Φ, the effective potential has only one minimum, inside the configuration; as Φ increases, also an external maximum and an external minimum appear. When Φ increases again, the internal minimum and the external maximum disappear and only the external minimum remains. This

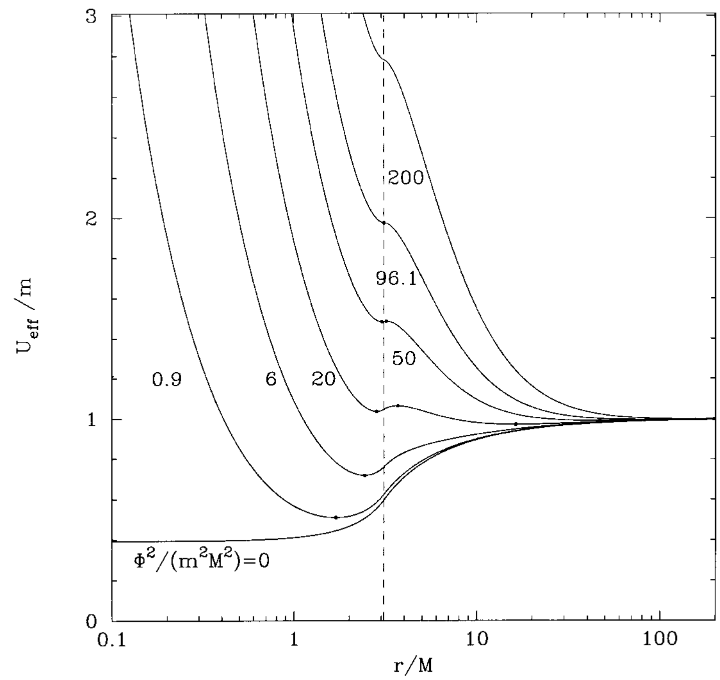

Fig. 2. – U_{eff}/m in the case of a fluid sphere with constant energy density and $R = 3.1M$, for selected values of $\Phi^2/(m^2M^2)$. For low values of Φ^2 the effective potential has only one minimum, inside the configuration; as Φ^2 increases, also an external maximum and an external minimum appear. When Φ^2 increases again, the internal minimum and the external maximum disappear and only the external minimum remains. Dots show circular orbits.

means that, also in this second case, circular orbits exist for every value of r, but not all of them are stable (those with the radius from R to $6M$ are unstable).

The third case (fig. 3) is the most interesting: for low values of the angular momentum the effective potential has only one minimum, inside the configuration; as Φ increases, also an external maximum and an external minimum appear; but in this case there are allowed circular orbits only for values of r which satisfy one of the two following conditions:

$$r > 3M \quad \text{(orbits in the exterior of the fluid sphere)},$$

$$\frac{r}{M} < \left(\frac{R}{M}\right)^{3/2} \sqrt{\frac{1}{2}\left[1 - \frac{1}{9(1-2M/R)}\right]} \quad \text{(orbits in the interior of the fluid sphere)}.$$

Clearly the problem of a fluid sphere with constant energy-density is only propaedeutic to the study of Einstein's clusters, in order to familiarize with the novel problem of circular orbits in the interior of a spherically symmetric relativistic configuration. However, quite apart from the analogies, there are differences from the Schwarzschild case, where the system is supposed to be a perfect fluid with constant energy density and pressure increasing inward. In the case of Einstein's

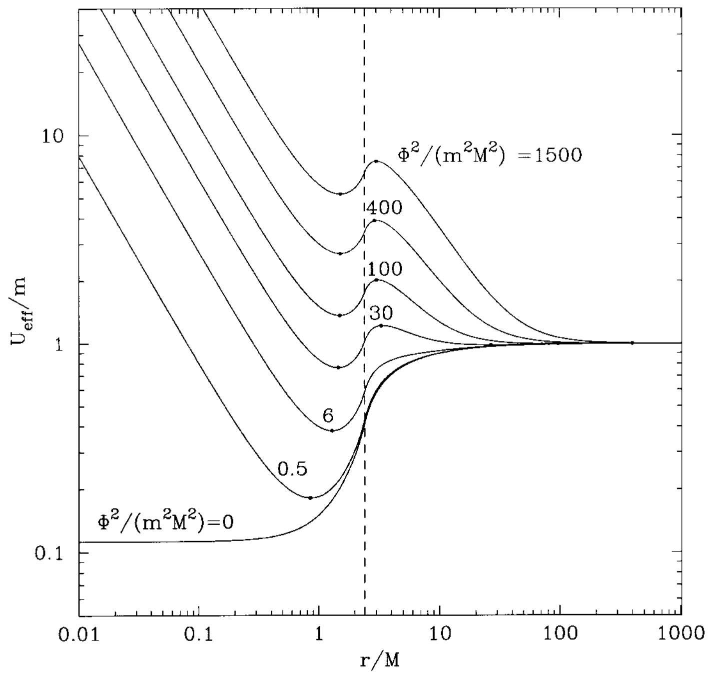

Fig. 3. – U_{eff}/m in the case of a fluid sphere with constant energy density and $R = 2.4M$, for selected values of $\Phi^2/(m^2 M^2)$. For low values of Φ^2 there is only an interior minimum. As Φ^2 increases, also an external maximum and an external minimum appear. No circular orbits are allowed for $1.52M < r < 3M$. Dots show circular orbits.

clusters, which are collisionless systems centrifugally supported, the radial pressure is identically zero.

3. – Einstein's clusters

In this model we can choose arbitrarily the number density of the particles or, equivalently, the energy density of the cluster.

In turn, for consistency, at every point r at which the particle density is different from zero the metric must allow the existence of circular orbits.

Since in the outermost layer of the system the gravitational field is given by the exterior Schwarzschild metric, the radius of an Einstein's cluster can never be less than $3M$.

In his original model (1939) Einstein used isotropic spherical coordinates. We reformulate the same problem in ordinary spherical coordinates.

The metric describing a generic static and spherically symmetric gravitational field, either in the interior or in the exterior of the matter distribution, can then be expressed in the form given by eq. (3), where now $a(r)$ and $b(r)$ are two functions of the radial coordinate r that must be determined solving the field equations.

The energy-momentum tensor has to be constructed by imposing that all the particles constituting the system move on the circular geodesics of the metric determined by their global gravitational field.

We have, for the only non-vanishing components,

$$T_t^t = \tilde{\varrho} = n(r)\, m a^{-1/2}\left(1 - \frac{r}{2}\frac{b'}{b}\right)^{-1/2},$$

$$T_\theta^\theta = T_\varphi^\varphi = -\frac{r}{4}\frac{b'}{b} n(r)\, m a^{-1/2}\left(1 - \frac{r}{2}\frac{b'}{b}\right)^{-1/2}.$$

For simplicity, we assume that all the particles constituting the cluster have the same mass m, and we call $\tilde{\varrho}$ the energy density of the cluster measured in the local inertial frame in which the center of mass of the considered volume element is at rest.

The non-trivial field equations are

$$\frac{b'}{b} = \frac{a-1}{r},$$

$$\frac{1}{r}\frac{a'}{a} + \frac{a}{r^2} - \frac{1}{r^2} = 8\pi\, n(r)\, m a^{1/2}\left(\frac{3}{2} - \frac{a}{2}\right)^{-1/2}.$$

In Einstein's clusters the necessary condition for the existence of circular orbits, and hence the self-consistency condition for the model, is

$$a(r) < 3 \iff r > 3\mathscr{M}(r) \tag{5}$$

(where $\mathscr{M}(r)$ is the total mass of the cluster included in the radius r).

It can be derived from the causality principle, using the explicit expression of the metric, the geodesic equations and the field equations.

We analyze three special cases:

a) $\tilde{\varrho}(r) = \text{const}$,

b) $n(r) = \text{const}$,

c) $n(r) = n(0)[1 - \exp[1 - R/r]]$

with particular attention to the stability of the circular orbits on which the particles constituting the clusters move.

An analytical stability condition for circular orbits can be easily determined by imposing the second derivative of the effective potential to be positive.

Using expression (4) for the effective potential and the field equations, it is direct to obtain

$$\left.\frac{\mathrm{d}^2 U_{\mathrm{eff}}}{\mathrm{d}r^2}\right|_{\Phi^2 = \Phi^2_{\mathrm{circ}}} > 0 \iff a'(r) > -\frac{[a(r)-1][3-a(r)]}{r}. \tag{6}$$

It can be also demonstrated that this condition is equivalent to these other two:

$$\frac{\mathrm{d}\Phi^2_{\mathrm{circ}}}{\mathrm{d}r} > 0 \iff \frac{\mathrm{d}E_{\mathrm{circ}}}{\mathrm{d}r} > 0, \tag{7}$$

where $\Phi^2_{\rm circ}$ and $E_{\rm circ}$ are the relativistic generalization of the angular momentum and of the energy of particles moving on circular orbits with radius r.

The explicit expressions of $\Phi^2_{\rm circ}$ and $E_{\rm circ}$ are

$$\Phi^2_{\rm circ} = m^2 r^2 \left[\frac{a(r) - 1}{3 - a(r)} \right],$$

$$\frac{E_{\rm circ}}{m} = \sqrt{\frac{2b(r)}{3 - a(r)}} .$$

If all orbits are stable then, from eq. (7), their energy and their angular momentum always increase with r. If the energy and the angular momentum do not increase monotically outward, then there must exist necessarily some unstable orbits.

We now consider the three special cases.

3·1. *Cluster with constant energy density.* – In this case, due to the simplicity of the hypothesis, it is possible to integrate analytically the field equations.

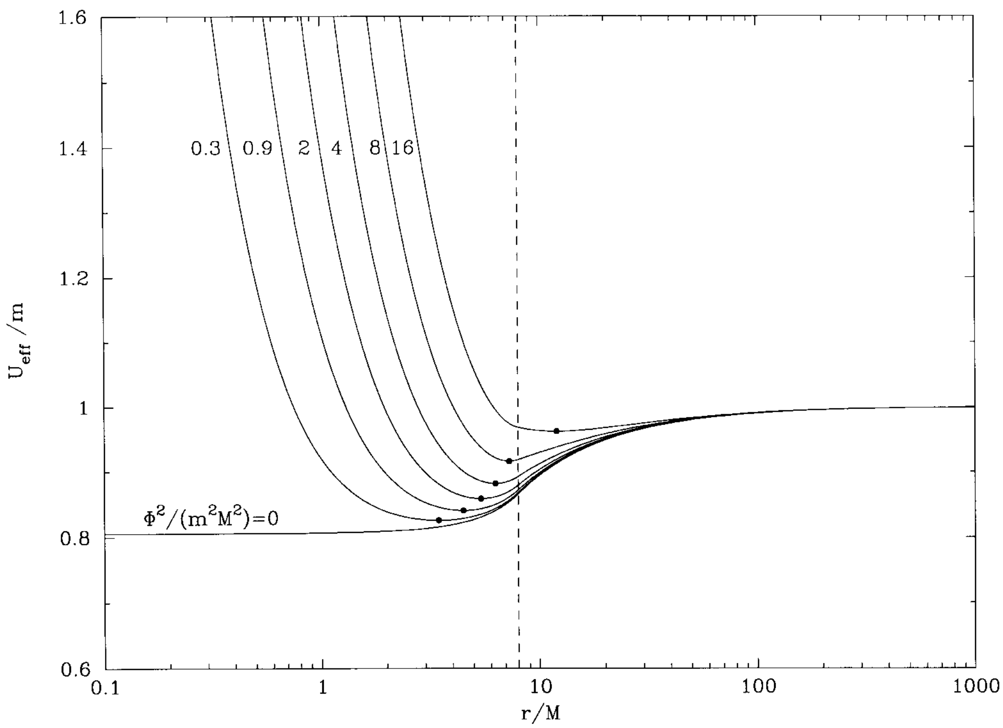

Fig. 4. – $U_{\rm eff}/m$ in the case of an Einstein's cluster with constant energy density and $R = 8M$, for selected values of $\Phi^2/(m^2M^2)$. Dots show circular orbits, all stable either in the interior or in the exterior of the configuration.

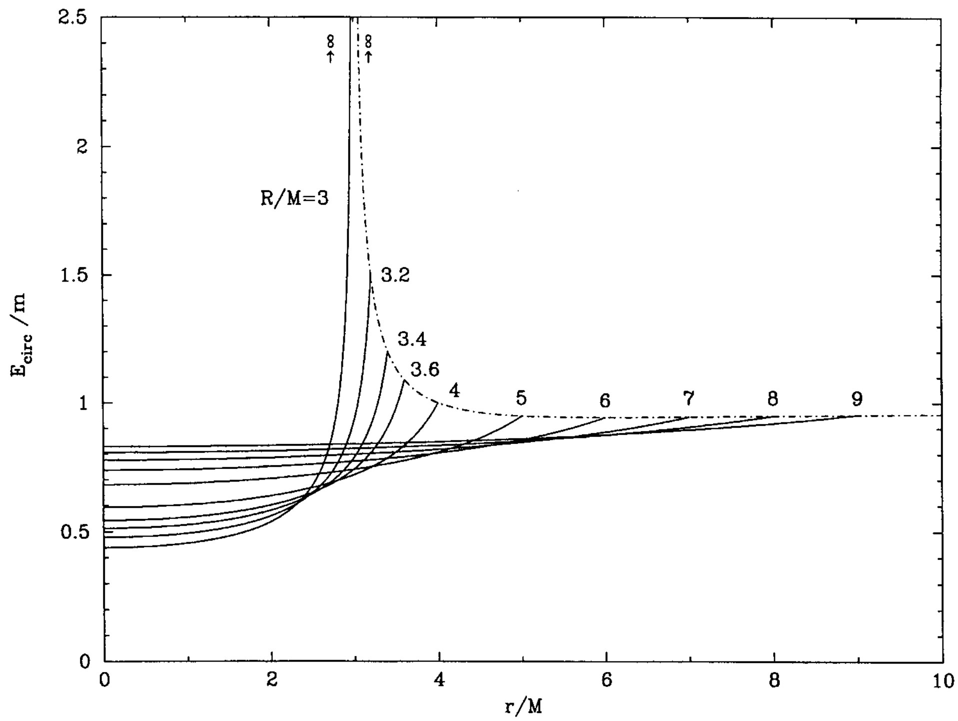

Fig. 5. – Energies of the circular orbits in units of the particle's mass, in the case of Einstein's clusters with $\tilde{\varrho} = \text{const}$ and selected values of R/M. If $R > 6M$ the energy always increases with r, either in the interior or in the exterior of the configuration (this means that circular orbits are all stable); while if $R < 6M$ there is a region of instability in the exterior of the configuration, corresponding to the decreasing of E_{circ}/m.

We find

$$a(y) = \left(1 - \frac{2Mr^2}{R^3}\right)^{-1},$$

$$b(y) = \left(1 - \frac{2M}{R}\right)^{3/2}\left(1 - \frac{2Mr^2}{R^3}\right)^{-1/2}.$$

With the only limitation that $R > 3M$, circular orbits are allowed for every value of r, either in the exterior or in the interior of the cluster, while their stability depends critically on the value of R/M.

It is convenient to distinguish two classes: models with $R > 6M$ and models with $3M < R < 6M$.

If $R > 6M$ all circular orbits are stable (see fig. 4 and fig. 5), so all the particles constituting the cluster move on circular stable orbits, and also in the exterior of the configuration every circular orbit allowed to a test-particle is stable.
If $3M < R < 6M$ all particles constituting the cluster move on stable circular orbits, but in the exterior of the configuration there is a region of instability ($R < r < 6M$), see, *e.g.*, fig. 6 and fig. 5.

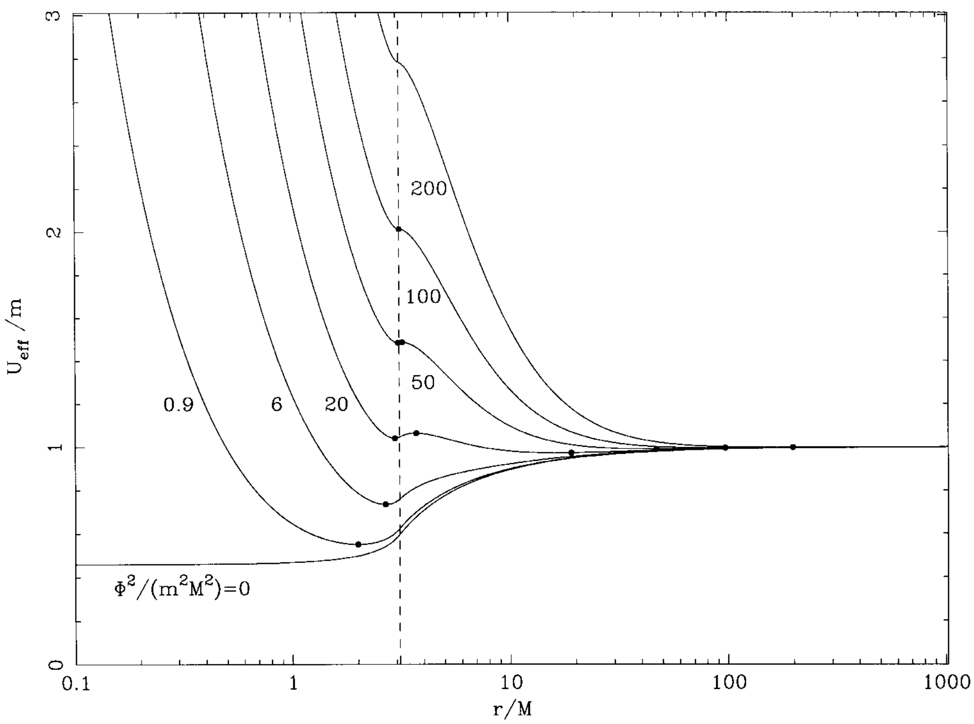

Fig. 6. – U_{eff}/m in the case of an Einstein's cluster with constant energy density and $R = 3.1M$, for selected values of $\Phi^2/(m^2M^2)$. In the interior of the configuration circular orbits are all stable, while in the exterior there is a region of instability.

3·2. *Cluster with constant particle density.* – In this case the analytical solution of Einstein's equations is unknown, but it is easy to integrate them numerically.

The results are qualitatively analogous to those obtained in the previous case, with only small quantitative changes in the binding energy.

3·3. *Cluster with an exponential decreasing particle density.* – Also in this case we proceed with numerical integration.

Circular orbits are allowed for every value of r (if $R > 3M$) and again it is convenient to distinguish between clusters with $R > 6M$ and clusters with $3M < R < 6M$.

If $R > 6M$, as in the previous two cases, all circular orbits are stable; while, if $3M < R < 6M$, the clusters have a central nucleus of particles which move on stable circular orbits and an external layer of particles which move on unstable orbits. In the exterior of the configuration there is a region of instability ($R < r < 6M$), while if $r > 6M$ all circular orbits are stable again (see fig. 7*a*)-*b*) and fig. 8).

4. – Classification of Einstein's clusters

On the basis of the analysis of the stability of circular orbits it is possible to divide Einstein's clusters in three classes: stable, unstable and metastable models.

By the term "stable models" we mean Einstein's clusters constituted by particles

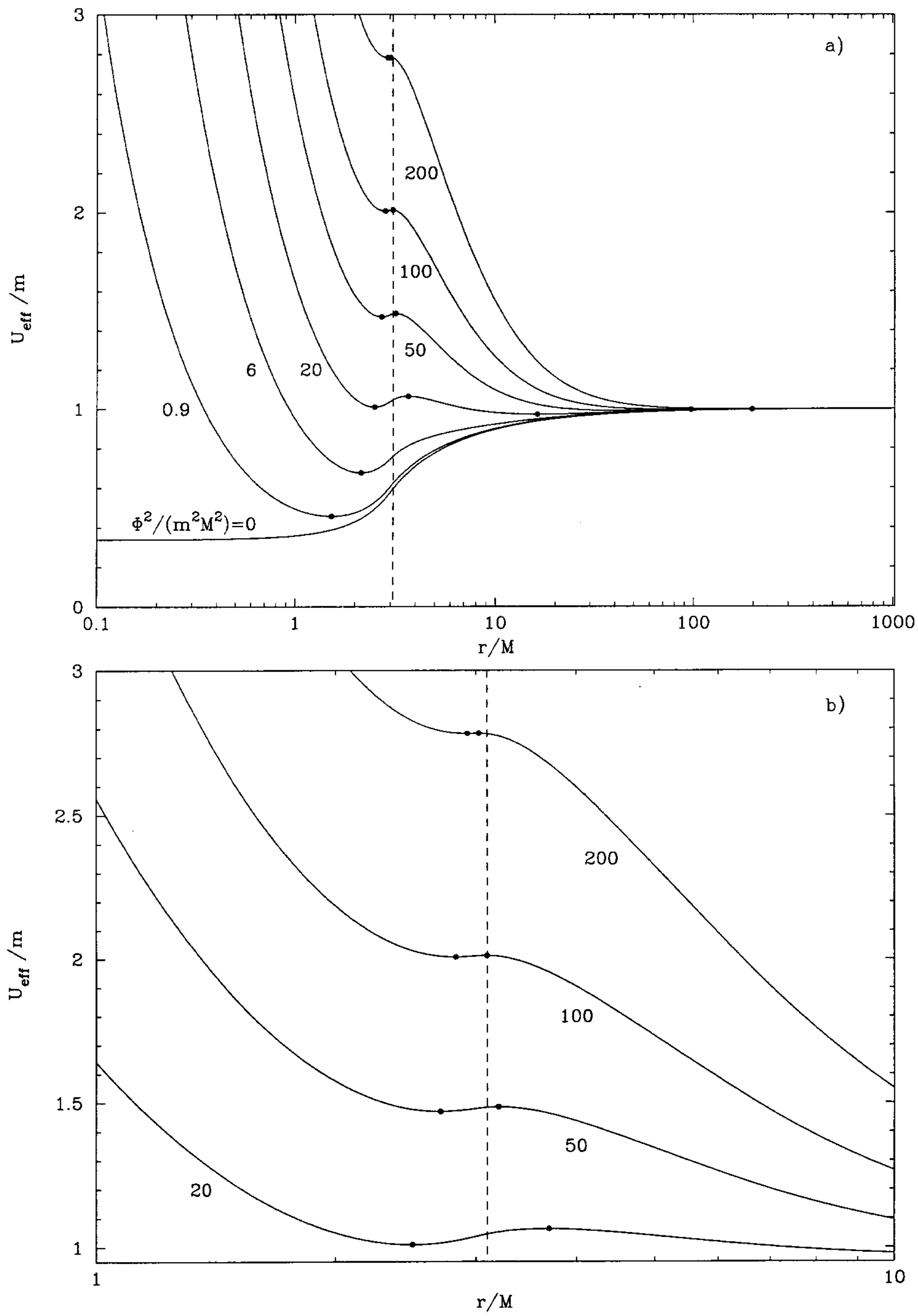

Fig. 7. – *a*) U_{eff}/m in the case of an Einstein's cluster with $n(r) = n(0)[1 - \exp[1 - R/r]]$ and $R = 3.1M$, for selected values of $\Phi^2/(m^2M^2)$. *b*) Detail of *a*) which shows the instability of circular orbits in the outermost layer of an Einstein's cluster with $n(r) = n(0)[1 - \exp[1 - R/r]]$ and $R = 3.1M$.

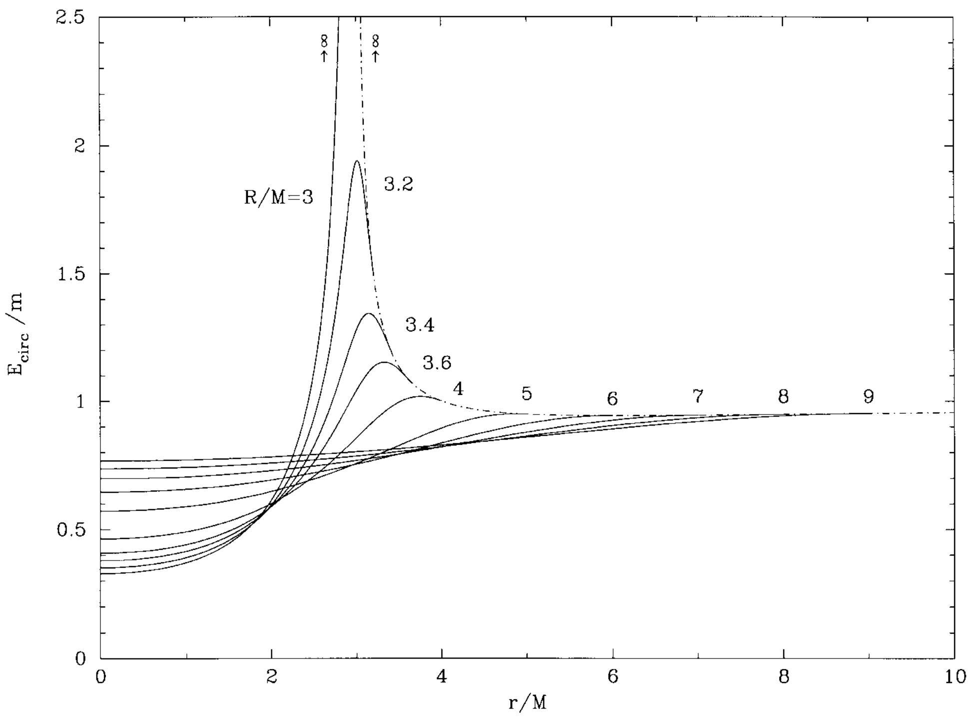

Fig. 8. – Energies of the circular orbits in units of the particle's mass, in the case of Einstein's clusters with $n(r) = n(0)[1 - \exp[1 - R/r]]$ and selected values of R/M. If $R > 6M$ the energy always increases with r and this means that circular orbits are all stable; while, if $R < 6M$, the energy does not increase monotically outward, and this means that circular orbits in the outermost layer of the cluster and in an external region are unstable.

which move all on stable circular orbits and for which the conditions (7) are verified for every value of r, either in the interior or in the exterior of the configuration. This means that every allowed circular orbit corresponds to an absolute minimum of the effective potential.

Stable configurations have necessarily $R > 6M$.

Examples of this kind of Einstein's clusters are models with $\tilde{\varrho}(r) = \text{const}$ or $n(r) = \text{const}$, or $n(r) = n(0)[1 - \exp[1 - R/r]]$ (and, obviously, $R > 6M$). See, *e.g.*, fig. 4 and fig. 5.

On the other hand, if some particles constituting an Einstein's cluster move on unstable circular orbits, then the configuration itself is unstable and it is characterized by the fact that

$$\frac{\mathrm{d}\Phi^2_{\text{circ}}}{\mathrm{d}r} < 0\,, \qquad \frac{\mathrm{d}E_{\text{circ}}}{\mathrm{d}r} < 0$$

in an internal region of the cluster.

A perturbation of an unstable model can produce the dispersion of a part of the system or a modification of the equilibrium configuration.

Examples of unstable models are Einstein's clusters with $n(r) = n(0)[1 - \exp[1 - R/r]]$ and $R < 6M$ (fig. 7*a*)-*b*) and fig. 8).

Finally, we define metastable models Einstein's clusters for which the conditions (7) are verified in every internal point, but are not verified in an exterior region adjacent to the cluster.

This means that some of the particles constituting the cluster move on stable circular orbits which correspond to a relative minimum of the effective potential, because it presents also a maximum and another minimum, both external to the configuration.

Metastable models have always $3M < R < 6M$.

We obtain examples of this kind of Einstein's clusters supposing $\tilde{\varrho}(r) = \text{const}$ or $n(r) = \text{const}$ or $n(r) = n(0)[1 - \exp[1 - R/r]]$ and, obviously, $3M < R < 6M$. (See, *e.g.*, fig. 6 and fig. 5.)

The perturbation of a metastable model produces an evaporation of the particles belonging to the outermost layer, because stable orbits with $r \simeq R$ become unstable. Such process is self-sustained: once the evaporation starts, it propagates inward.

5. – Einstein's clusters with arbitrarily large values of the central gravitational redshift

We turn now to the issue of the limits of the central gravitational redshift in self-gravitating clusters, which has been the subject of study in many specific examples [2, 7-13, 15-18, 24].

We have found that the approach to an infinite redshift is in general not reachable through a sequence of stable clusters. There are, however, special energy density distributions in which this phenomenon is possible.

In particular, we consider Einstein's clusters with the following energy density distribution:

$$\tilde{\varrho} = \tilde{\varrho}_c \frac{1}{1 + (r/r_0)^2} \qquad \text{for } r \leqslant R ,$$

where $\tilde{\varrho}_c$ and r_0 are the central density and the radius of the cluster's core.

Fixing the value of R/M and decreasing the radius of the core (r_0/M) it is possible to obtain arbitrarily large values of the central gravitational redshift.

For the above distribution we have that the mass included in the radius r is given by

$$\mathscr{M}(r) = 4\pi \tilde{\varrho}_c r_0^3 \left[\frac{r}{r_0} - \arctan\left(\frac{r}{r_0}\right)\right] \Rightarrow M = 4\pi \tilde{\varrho}_c r_0^3 \left[\frac{R}{r_0} - \arctan\left(\frac{R}{r_0}\right)\right] \Rightarrow$$

$$\Rightarrow \tilde{\varrho}_c = \frac{M}{4\pi r_0^2 [R - r_0 \arctan(R/r_0)]} .$$

The energy density distribution can then be rewritten as

$$\tilde{\varrho}(r) = \frac{M}{4\pi [R - r_0 \arctan(R/r_0)]} \frac{1}{(r^2 + r_0^2)} . \tag{8}$$

From the above formula it is evident that

$$\lim_{r_0 \to 0} \tilde{\varrho}(r) = \frac{M}{4\pi R} \frac{1}{r^2} .$$

It is this asymptotic behaviour of the energy density distribution in the limit $r_0 \to 0$ that leads to an infinite central gravitational redshift. We can see this explicitly.

In the hypothesis (8) the field equations can be integrated analitically; the result is

$$a(r) = \frac{1}{1 - \frac{2\mathscr{M}(r)}{r}} = \frac{1}{1 - \frac{8\pi\tilde{\varrho}_c r_0^2}{r}[r - r_0 \arctan(r/r_0)]} =$$

$$= \left[1 - \frac{2M}{r} \frac{r - r_0 \arctan(r/r_0)}{R - r_0 \arctan(R/r_0)}\right]^{-1},$$

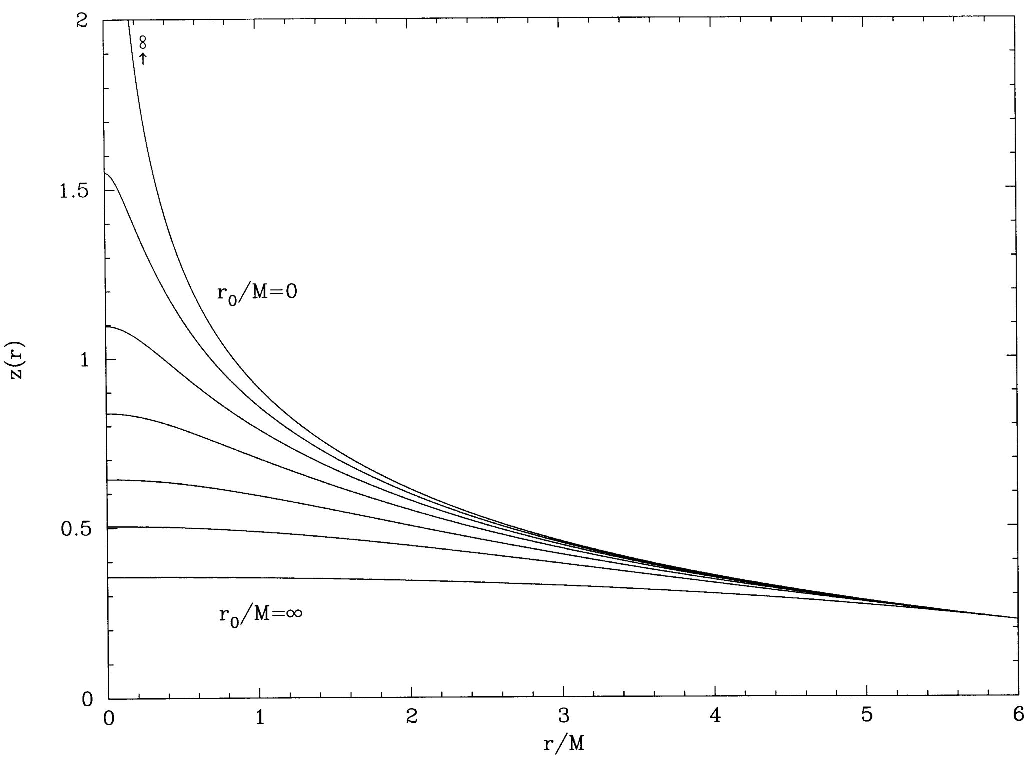

Fig. 9. – Gravitational redshift in the case of an Einstein's cluster with $\tilde{\varrho}/\tilde{\varrho}_c = [1 - (r/r_0)^2]^{-1}$ and $R = 6M$, for the following values of r_0/M: 0.00, 10^{-1}, 0.25, 0.50, 1.00, 2.00, ∞.

$$b(r) = \left(1 - \frac{2M}{R}\right) \exp\left[-\int_r^R \frac{a-1}{r}\,\mathrm{d}r\right] = \frac{1-2M/R}{\exp[I(r)]}\,,$$

$$I(r) = \int_r^R \left\{\frac{1}{r}\left[1 - \frac{2M}{r}\,\frac{r - r_0\arctan(r/r_0)}{R - r_0\arctan(R/r_0)}\right]^{-1} - \frac{1}{r}\right\}\mathrm{d}r\,.$$

In the limit $r_0 \to 0$, $a(r)$ and $b(r)$ assume the form

$$\lim_{r_0\to 0} a(r) = \left(1 - \frac{2M}{R}\right)^{-1},$$

$$\lim_{r_0\to 0} b(r) = \frac{1-2M/R}{\exp\left[\int_r^R \frac{2M/R}{r(1-2M/R)}\,\mathrm{d}r\right]} = \left(1 - \frac{2M}{R}\right)\left(\frac{r}{R}\right)^{\frac{2M}{R-2M}}.$$

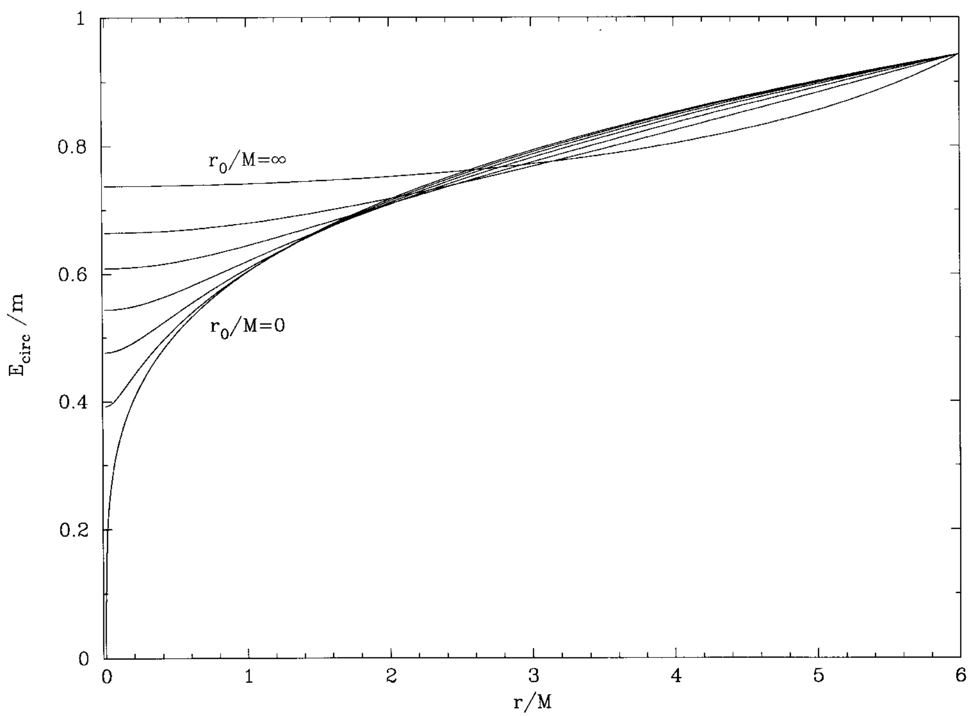

Fig. 10. – Energies of the circular orbits in units of the particle's mass, in the case of Einstein's clusters with $\tilde{\varrho}/\tilde{\varrho}_c = [1-(r/r_0)^2]^{-1}$, with $R = 6M$ and with the following values of r_0/M: 0.00, 10^{-1}, 0.25, 0.50, 1.00, 2.00, ∞.

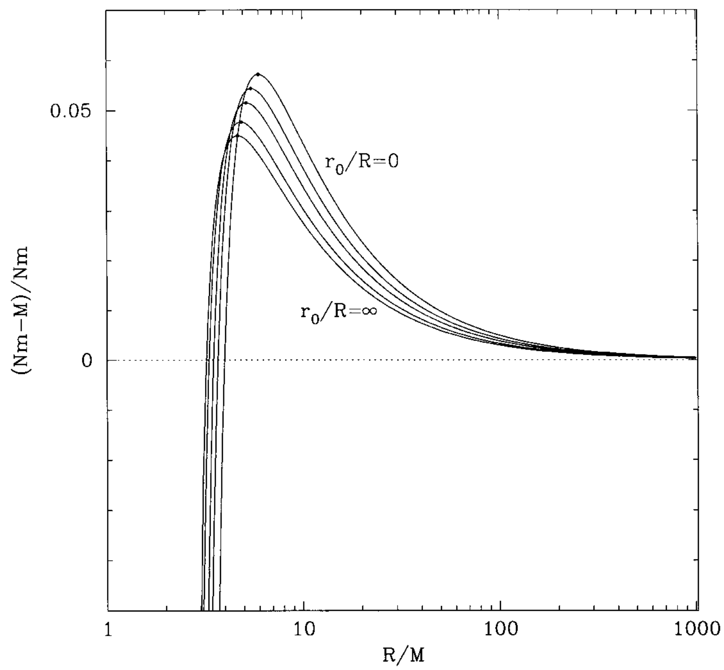

Fig. 11. – Fractional binding energy *vs.* R/M in the case of Einstein's clusters with $\tilde{\varrho}/\tilde{\varrho}_c = [1-(r/r_0)^2]^{-1}$, for the following values of r_0/R: 0.00, 0.05, 0.20, 0.50, ∞.

From the above expression it is evident that, whatever is the value of R/M, in the limit $r_0 \to 0$ the central gravitational redshift becomes infinite (fig. 9):

$$\lim_{r_0 \to 0} z_c = \lim_{r_0 \to 0} \frac{1}{\sqrt{b(0)}} = \infty \; .$$

The most relevant aspect of this kind of Einstein's clusters is that they are strictly stable all the way up to the approach of an infinite central gravitational redshift. In fact we can prove that the conditions (5) and (6) for the existence and the stability of circular orbits are always fulfilled, independently of the values of R/M and r_0/R; it then follows that, if $R > 6M$, they are strictly stable (see the definition of stability given in sect. **4**).

Figure 10 shows, in specific examples with $R = 6M$ and selected values of r_0/M, that the energy of circular orbits always increases monotically outward; in accord with the condition (7), this fact can be considered another prove of the stability of these configurations.

Another significant quantity is the fractional binding energy of the cluster: the relative decrease of its total energy when it contracts from an infinite radius to the radius R. Figure 11 shows how it varies with varying R/M, for fixed values of r_0/R.

We like to stress that the situation decribed above (existence of families of Einstein's clusters stable all the way up to an infinite central gravitational redshift) is far from general. If one considers, *e.g.*, a distribution of constant energy density, the maximum central gravitational redshift compatible with stability is $z_c = 1.35538$.

6. – Conclusions

The consideration of Einstein's clusters, due to the generality and simplicity of the model, is of the greatest relevance in the understanding of relativistic stellar clusters.

On the one hand Einstein's clusters give direct and straightforward examples of clusters reaching infinite central gravitational redshift, which have been the topic of an active debate for many years.

On the other hand, they show the existence of a new phenomenon: the metastable clusters which, to our knowledge, have not yet been addressed in the literature by examples with specific distribution functions.

The forthcoming effort will consist in generalizing these results to axially symmetric configurations, non-rotating, following the work of Morgan and Morgan [14], and endowed also with rotation, following the work of Bardeen and Wagoner [21, 22].

The possible relevance for active galactic nuclei will be analyzed elsewhere.

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