Gravitational radiation emitted when a mass falls onto a compact star (*)

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Summary. — In this paper we study the energy spectrum related to the axial perturbations of a compact star when a particle falls spiralling onto it. We find that both slowly-damped quasi-normal modes and strongly damped *w*-modes are excited, and that a part of the energy in the process is associated to these *w*-modes. Our analysis will show a substantial difference between the energy spectrums of compact stars and black holes.

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1. – Introduction

The process of capture of masses by black holes and stars can be investigated, in the hypotesis that the infalling mass m_0 is smaller than that of a black hole or of a star M, with the technics of first-order perturbations of the background geometry. The case involving black holes, and static black holes in particular, was studied in full in the literature. Einstein's perturbed equations split into two separated sets of equations that can be reduced to two second-order inhomogeneous wave equations with real (and different) potential barriers and different source terms describing the radial part of axial and polar perturbations (Regge and Wheeler, 1957 [1], Zerilli, 1970 [2]) ([1]). The source term for these equations comes out from the stress-energy tensor of the mass m_0 which is assumed to fall along a geodesic of the unperturbed Schwarzschild space-time. A particle can fall following a radial or a spiralling trajectory, and an analysis of

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([1]) The *axial* and *polar* perturbations used in this paper and introduced by Chandrasekhar [3] correspond, in our gauge, to *odd* and *even* perturbations in the Regge-Wheeler gauge. Under a parity transformation the *axial* perturbations transform like $(-1)^{(l+1)}$ while the *polar* perturbations transform like $(-1)^l$.

the source term shows that for polar perturbations it is different from zero in both cases and for axial perturbations it is non-null only in the second case. It means that axial perturbations can be excited only if the particle has an initial angular momentum. The inhomogeneous equations describing the problem can be integrated numerically [4-13] and waveforms, energies and spectra of the emitted radiation determined.

The case of a particle falling onto a star was never studied before. The corresponding equations are more complicated than those of the black-hole case for the presence of the fluid composing the star, but they again split (in the case of a non-rotating star) into two distinct sets, the polar one and the axial one. The polar equations inside the star are a fifth-order linear system [14, 15] and outside they can be reduced to the same second-order wave equation found for polar perturbation of a black hole while the equations for the axial perturbations are simpler, and can be reduced both inside and outside the star to a single second-order equation with a central potential barrier depending on how the energy density and the pressure are distributed inside the star in the imperturbed configuration. Nevertheless, the solution of the perturbed equations for stars and black holes is subject to different boundary conditions since for black holes we have to impose the condition that only ingoing radiation must be present at the horizon while for stars the required condition is that interior and exterior metric perturbations must be matched continuously at the boundary of the stars.

In this paper we will restrict our analysis to the study of the axial perturbations excited by a spiralling particle onto a nonrotating very compact star. In our study, we will consider only stars with a uniform energy density distribution because in this model, although unrealistic, the effects of general relativity are stronger than in any other stellar model. In ref. [16] it was shown that if the radius of a compact star is smaller than $3M$, there exist slowly damped quasi-normal modes which are "trapped" by the potential barrier generated by the space-time curvature and these modes become more numerous as the star becomes more compact. Further investigations of the same model proved the existence of strongly damped w-modes [17-20].

The question of whether these modes can be excited arises as well as the one of possible differences between the spectrums of black holes and very compact stars.

2. – The unperturbed configuration

The metric describing a non-rotating star with uniform energy density $\varepsilon = \text{const}$ and pressure p, has the standard form [21]

$$ds^2 = e^{2\nu} dt^2 - e^{2\mu_2} dr^2 - r^2 (d\vartheta^2 + \sin^2\vartheta \, d\varphi^2), \tag{1}$$

with

$$\begin{cases} e^{2\nu} = \dfrac{(3y_1 - y)^2}{4}, & e^{-2\mu_2} = \left(1 - \dfrac{2\varepsilon r^2}{3}\right), \\ y = \left(1 - \dfrac{2\varepsilon r^2}{3}\right)^{1/2}, & y_1 = \left(1 - \dfrac{2\varepsilon r_1^2}{3}\right)^{1/2}, \\ m(r) = \dfrac{\varepsilon r^3}{3}, & p = \dfrac{\varepsilon(y - y_1)}{3y_1 - y}. \end{cases} \tag{2}$$

while the stress-energy tensor of the perfect fluid composing the star is

(3) $$T_{\mu\nu(f)} = (\varepsilon + p)\, u_\mu u_\nu - p g_{\mu\nu}, \qquad u_t = e^\nu, \quad u_\varphi = u_r = u_\vartheta = 0 \,.$$

By imposing the continuity of the metric at the boundary of the star $r = r_1$,

(4) $$e^{2\nu}_{(r=r_1)} = e^{-2\mu_2}_{(r=r_1)} = 1 - 2M/r_1,$$

the exterior metric reduces to the Schwarzschild metric.

From eqs. (2) it follows that we must have

(5) $$y_1 = \left(1 - \frac{2\varepsilon r_1^2}{3}\right)^{1/2} > \frac{1}{3}, \qquad r_1/M > 2.25 \,,$$

a result due to Schwarzschild. Our stellar models can be labelled by the parameter (r_1/M).

The stress-energy tensor $T_{\mu\nu(\mathrm{p})}$ of a particle falling along a geodetic $(T(t), R(t), \theta(t), \phi(t))$ of the imperturbed space-time is

(6) $$T^{\mu\nu}_{(\mathrm{p})} = m_0 \gamma \frac{\mathrm{d}z^\mu}{\mathrm{d}t} \frac{\mathrm{d}z^\nu}{\mathrm{d}t} \frac{\delta(r - R(t))}{r^2} \delta(\Omega - \Omega(t)) \qquad \left(\gamma = \frac{\mathrm{d}T(t)}{\mathrm{d}\tau}\right).$$

As in every central field the motion takes place on a plane and, for particles with angular momentum different from zero, we can assume as plane of the orbit the equatorial one characterized by $\theta = \frac{\pi}{2}$.

3. – The choice of the gauge

According to the theory, the expansion of a rank-2 tensor in tensor harmonics of a particular degree l can be obtained once fixed the basis tensors (see, for example, [22]). For a symmetric tensor as the one describing the general (non-axisymmetric) perturbation $h_{\mu\nu}$ of the background metric (1) these harmonics can be expressed in simpler matrices given (apart from some errors) by Zerilli [2] and involving spherical harmonics and their derivatives. By choosing the coefficient functions of the expansion in such a way as to agree with the notation of Regge-Wheeler [1], the tensor can be written as follows [2]:

(7) $$\begin{aligned} h_{lm} = {} & e^{2\mu} H_{0lm}(r,t)\, \mathbf{a}^{(0)}_{lm} + \sqrt{2} H_{1lm}(r,t)\, \mathbf{a}^{(1)}_{lm} + e^{2\nu_2} H_{2lm}(r,t)\, \mathbf{a}_{lm} + \\ & + \frac{\sqrt{2}\, C_1}{r} h_{0lm}(r,t)\, \mathbf{b}^{(0)}_{lm} - \frac{\sqrt{2}\, C_1}{r} h_{1lm}(r,t)\, \mathbf{b}_{lm} + G_{lm}(r,t) \frac{\sqrt{2}\, C_2}{2} \mathbf{f}_{lm} + \\ & + \sqrt{2}\left(K_{lm}(r,t) - \frac{G_{lm}(r,t)}{r} l(l+1)\right) \mathbf{g}_{lm} - \frac{l}{r^2} \sqrt{2}\, C_2 h_{2lm}(r,t)\, \mathbf{d}_{lm} - \\ & - \frac{l}{r} \sqrt{2}\, C_1 h_{0lm}(r,t)\, \mathbf{c}^{(0)}_{lm} + \frac{l}{r} \sqrt{2}\, C_1 h_{1lm}(r,t)\, \mathbf{c}_{lm}. \end{aligned}$$

where

$$C_1 = \frac{1}{\sqrt{l(l+1)}}\,, \qquad C_2 = \frac{1}{\sqrt{l(l+1)(l-1)(l+2)}}\,. \tag{8}$$

By using a gauge in such a way that when $m = 0$ (axisymmetric perturbations) the polar part coincides with the one in Chandrasekhar gauge while the axial part coincides with the one in Regge-Wheeler gauge and by supposing that the time dependence of the perturbations is given by $e^{\imath\omega t}$, *i.e.* by Fourier-expanding all functions with the convention

$$M(\omega, r) = \int_{-\infty}^{+\infty} M(t, r)\, e^{-\imath\omega t}\,\mathrm{d}t\,, \tag{9}$$

the first-order, non-axisymmetric perturbations exicited by a particle spiralling onto a star can be described by the following perturbed metric:

$$\begin{aligned} \mathrm{d}s^2 = {} & e^{2\nu}[1 + H_0(r)\, Y_{lm}\, e^{\imath\omega t}]\,\mathrm{d}t^2 - e^{2\mu_2}[1 - H_2(r)\, Y_{lm}\, e^{\imath\omega t}]\,\mathrm{d}r^2 - \\ & -(e^{2\psi} - H_{11}\, e^{\imath\omega t})\,\mathrm{d}\varphi^2 - (e^{2\mu_3} - H_{33}\, e^{\imath\omega t})\,\mathrm{d}\vartheta^2 + \\ & + h_0(r)\sin\vartheta\, Y_{lm,\,\vartheta}\, e^{\imath\omega t}\,\mathrm{d}t\,\mathrm{d}\varphi - h_0(r)\,\frac{1}{\sin\vartheta}\, Y_{lm,\,\varphi}\, e^{\imath\omega t}\,\mathrm{d}t\,\mathrm{d}\vartheta + \\ & + h_1(r)\sin\vartheta\, Y_{lm,\,\vartheta}\, e^{\imath\omega t}\,\mathrm{d}r\,\mathrm{d}\varphi - h_1(r)\,\frac{1}{\sin\vartheta}\, Y_{lm,\,\varphi}\, e^{\imath\omega t}\,\mathrm{d}r\,\mathrm{d}\vartheta + \\ & + G(r)(Y_{lm,\,\vartheta,\,\varphi} - \cot\vartheta\, Y_{lm,\,\varphi})\, e^{\imath\omega t}\,\mathrm{d}\varphi\,\mathrm{d}\vartheta\,, \end{aligned} \tag{10}$$

where

$$H_{11} = r^2[K(r)\sin^2\vartheta\, Y_{lm} + G(r)(Y_{lm,\,\vartheta,\,\vartheta} + \cos\vartheta\sin\vartheta\, Y_{lm,\,\vartheta})]\,, \tag{11}$$

$$H_{33} = r^2[K(r)\, Y_{lm} + G(r)\, Y_{lm,\,\vartheta,\,\vartheta}]\,. \tag{12}$$

4. – The equations governing the axial perturbations

Since, at the first order, *axial* and *odd* perturbations do not couple, the equations of the *axial* perturbations are the non-trivially zero axial components of Einstein equations. If we stop our analysis just before the particle crosses the surface of the star, the relevant Einstein equations are

$$\begin{cases} \delta G_{\varphi r} = 2\delta T_{\varphi r(\mathrm{f})} + 2\, T_{\varphi r(\mathrm{p})}\,, \\ \delta G_{\varphi\theta} = 2\delta T_{\varphi\theta(\mathrm{f})} + 2\, T_{\varphi\theta(\mathrm{p})}\,, \\ \delta G_{t\varphi} = 2\delta T_{r\varphi(\mathrm{f})} + 2\, T_{t\varphi(\mathrm{p})}, \end{cases} \tag{13}$$

where f denotes the fluid, p the particle, $T_{\mu\nu(\mathrm{p})} = 0$ inside the star and $T_{\mu\nu(\mathrm{f})} = 0$ outside the star, that give

$$e^{-2\nu}\left(-\omega^2 h_1 - h_{0,r} + \frac{2\imath\omega}{r} h_0\right) + \frac{2n}{r^2} h_1 = 4\imath r C_1 Q\,, \tag{14}$$

$$-\imath\omega e^{-2\nu} h_0 - e^{-2\mu_2}[(\mu_{2,r} + \nu_{,r}) h_1 + h_{1,r}] = -4\imath r^2 C_2 D\,, \tag{15}$$

$$e^{-2\mu_2}\Bigg[h_{0,r,r} - (\mu_{2,r} + \nu_{,r}) h_{0,r} - \imath\omega h_{1,r} - \imath\omega\left(\frac{2}{r} - \mu_{2,r} - \nu_{,r}\right) h_1 - \\ - \frac{2}{r}\left(\mu_{2,r} + \nu_{,r} - \frac{1}{r}\right) h_0\Bigg] - (l+1)(l-2)\frac{1}{r^2} h_0 = 4\imath r C_1 Q^{(0)} - 4(\varepsilon + p)\,\xi_{1,t}. \tag{16}$$

Here $\xi_{1,t} = e^{\nu}\delta u_1 \frac{1}{\sin\vartheta} Y_{lm,\vartheta}$,

$$D_{lm}(\omega, r) = 4\pi\frac{m_0 C_2 C_3 \tilde{L}^2\sqrt{2m}}{r^4}\left(\frac{\partial P_{lm}(\vartheta)}{\partial\vartheta}\right)\Bigg|_{(\vartheta=\pi/2)} \frac{e^{-\imath\omega T(r)} e^{-\imath m\phi(T(r))}}{\left[\frac{2M}{r} - \tilde{L}^2\frac{r-2M}{r^3}\right]^{+1/2}}\,,$$

$$Q_{lm}(\omega, r) = -4\pi\frac{\imath m_0 C_1 C_3 \tilde{L}\sqrt{2}}{r^2(r-2M)}\left(\frac{\partial P_{lm}(\vartheta)}{\partial\vartheta}\right)\Bigg|_{(\vartheta=\pi/2)} e^{-\imath\omega T(r)} e^{-\imath m\phi(T(r))},$$

$$Q^{(0)}_{lm}(\omega, r) = -4\pi m_0 C_1 C_3\frac{\tilde{L}}{r^3}\left(\frac{\partial P_{lm}(\vartheta)}{\partial\vartheta}\right)\Bigg|_{(\vartheta=\pi/2)} \frac{e^{-\imath\omega T(r)} e^{-\imath m\phi(T(r))}}{\left[\frac{2M}{r} - \tilde{L}^2\frac{r-2M}{r^3}\right]^{+1/2}}\,,$$

are the source terms obtained by developing the stress-energy tensor of the infalling particle in tensorial armonics, $\tilde{L}$ is its angular momentum normalized to the particle mass $\tilde{L} = L/m_0$,

$$C_3 = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}} \tag{17}$$

and P_{lm} are the Legendre polynomials.

To these equations we have to add the non-trivial components of the hydrodynamical equations $T^{\mu\nu}{}_{;\nu} = 0$. At the boundary of the star, the stress-energy tensor is given by the stress-energy tensor of the particle and by stress-energy tensor of the fluid composing the star. By differentiating eq. (16) with respect to time and making use of eqs. (14) (15), we find

$$\frac{\imath}{\sqrt{2}}\Bigg\{-(l+1)(l-2)\,C_2 e^{2\nu} D_{lm} + C_1 r\frac{\mathrm{d}}{\mathrm{d}t} Q^{(0)}_{lm} + \\ + C_1 e^{2(\nu-\mu_2)}\left[\frac{\mathrm{d}}{\mathrm{d}r}(rQ_{lm}) + (r\nu_{,r} - r\mu_{2,r} + 2)\,Q_{lm}\right]\Bigg\} = e^{\nu}\varepsilon\xi_{1,tt}. \tag{18}$$

It can be shown that eq. (18) coincides with the φ-component of the (axial) hydrodynamical equations written at the boundary

$$(T^{\mu\nu}{}_{(f)} + T^{\mu\nu}{}_{(p)})_{;\,\nu} = 0\,. \tag{19}$$

Let us come back to the two eqs. (14)-(15) and introduce the new variable

$$R(\omega,\, r) = \frac{e^{\nu-\mu_2}}{r}\, h_1(\omega,\, r)\,, \tag{20}$$

and the tortoise coordinate

$$r_* = \int_0^r e^{-\nu+\mu_2}\, \mathrm{d}r\,. \tag{21}$$

Equation (15) gives

$$h_0 = \frac{1}{\omega}\left[-\frac{4e^{2\nu} r^2 C_2}{\sqrt{2}}\, D_{lm} - i\,\frac{\mathrm{d}}{\mathrm{d}r_*}\,(rR)\right]. \tag{22}$$

This equation holds also in the interior of the star if we put $D_{lm} = 0$. By substituting (22) in eq. (14) and by making use of the unperturbed equations, we find the equations governing the perturbations inside the star:

$$\begin{cases} \dfrac{\mathrm{d}^2 R_{lm}}{\mathrm{d}r^2_*} + [\omega^2 - V_{\mathrm{int}}]\, R_{lm} = 0\,, \\ V_{\mathrm{int}} = (e^{2\nu}/r^3)[l(l+1)\, r + r^3(\varepsilon - p) - 6m(r)]\,, \end{cases} \tag{23}$$

and outside the star

$$\begin{cases} \dfrac{\mathrm{d}^2 R_{lm}}{\mathrm{d}r^2_*} + [\omega^2 - V_{\mathrm{ext}}]\, R_{lm} = S_{lm}\,, \\ V_{\mathrm{ext}} = (e^{2\nu}/r^3)[l(l+1)\, r - 6M]\,, \\ S_{lm} = -\dfrac{4\imath e^{\nu-\mu_2}}{\sqrt{2}}\left\{C_1 e^{2\nu} Q_{lm} + 2C_2 e^{2\nu} D_{lm} - \dfrac{C_2}{r}\,\dfrac{\mathrm{d}}{\mathrm{d}r}[r^2 e^{2\nu} D_{lm}]\right\}. \end{cases} \tag{24}$$

The dependence of the perturbation on the azimuthal parameter m is restricted to the source term of eq. (24) due to the infalling particle.

5. – The integration of the axial equations and the matching conditions

The equation governing the axial perturbations has the form

$$LR(r_*,\, \omega) = S(r,\, \omega)\, H(r_* - r_{1*})\,, \qquad 0 \leqslant r_* < \infty\,, \tag{25}$$

where L is the differential operator

$$L = \frac{\mathrm{d}^2}{\mathrm{d}r_*^2} + \omega^2 - V(r)\,. \tag{26}$$

5·1. *Solution inside the star.* – Firstly, we integrate the equation in the interior of the star for different values of the real frequency ω. For $r_* < r_{1*}$ (25) reduces to

$$LR(r_*, \omega) = 0 \tag{27}$$

with $V(r) = V_{(\mathrm{int})}(r)$.

Since the variable r_* is not explicitly known as a function of r, in place of eq. (23) it is convenient to integrate the corresponding equation in terms of r.

Near the origin the function $X_{lm} = rR_{lm}$ has the asymptotic expansion [15]

$$X_{lm} = r^{l+2} + \frac{1}{2(2l+3)}\left\{(l+2)\left[\frac{1}{3}(2l-1)\,\varepsilon_0 - p_0\right] - \omega^2 e^{2\nu}(r=0)\right\} r^{l+4} + \dots\,. \tag{28}$$

Starting with this expression and integrating up to the boundary r_1 of the star, we find the quantities

$$\begin{cases} r_1 X_{lm}(\omega, r_1) = a(\omega)\,, \\ [rX_{lm,}(\omega, r)]_{,r*}\big|_{(r=r_1)} = \left(1 - \dfrac{2M}{r}\right)[rX_{lm}(\omega, r)]_{,r*}\Big|_{(r=r_1)} = b(\omega) \end{cases} \tag{29}$$

and so, up to an arbitrary *complex* constant $\chi(\omega)$ (to be determined by matching the interior and the exterior solution) the amplitude of the perturbation and its derivative at the boundary:

$$\begin{cases} R_{lm}(\omega, r_1) = \chi(\omega) a(\omega)\,, \\ R_{lm,}(\omega, r)_{,r*}\big|_{(r=r_1)} = \chi(\omega)\, b(\omega)\,. \end{cases} \tag{30}$$

5·2. *Solution outside the star.* – Outside the star we must consider the equation

$$LR_{lm} = S_{lm} \tag{31}$$

with $V(r) = V_{(\mathrm{ext})}(r)$ an the initial conditions (30), *i.e.* the system

$$\begin{cases} LR_{lm} = S_{lm}\,, \\ R_{lm}(\omega, r_{1*}) = \chi(\omega)\, a(\omega)\,, \qquad r_{1*} \leqslant r_* < \infty\,, \\ R'_{lm}(\omega, r_{1*}) = \chi(\omega)\, b(\omega)\,. \end{cases} \tag{32}$$

The solution can be decomposed into two parts:

$$R_{lm}(\omega, r_*) = R_{1lm}(\omega, r_*) + R_{2lm}(\omega, r_*) \tag{33}$$

satisfying, respectively, the following systems:

$$(34)\qquad \begin{cases} LR_{1lm} = 0\,, \\ R_{1lm}(\omega,\, r_{1*}) = \chi(\omega)\, a(\omega)\,, \qquad r_{1*} \leqslant r_* < \infty\,, \\ R'_{1lm}(\omega,\, r_*)\big|_{(r_*=r_{1*})} = \chi(\omega)\, b(\omega) \end{cases}$$

and

$$(35)\qquad \begin{cases} LR_{2lm} = S\,, \\ R_{2lm}(\omega,\, r_{1*}) = 0\,, \qquad r_{1*} \leqslant r_* < \infty\,, \\ R'_{2lm}(\omega,\, r_*)\big|_{(r_*=r_{1*})} = 0\,. \end{cases}$$

The solution of the system (34) is a linear combination of the two independent solutions of the homogeneous equation, say $u(\omega,\, r_*)$ and $v(\omega,\, r_*)$,

$$(36)\qquad R_{1lm}(\omega,\, r_*) = \alpha(\omega)\, u(\omega,\, r_*) + \beta(\omega)\, v(\omega,\, r_*)\,,$$

where $\alpha(\omega)$ and $\beta(\omega)$ are *complex* coefficients to be determined in such a way as to satisfy the *complex* boundary conditions

$$(37)\qquad \begin{cases} \alpha(\omega)\, u(\omega,\, r_{1*}) + \beta(\omega)\, v(\omega,\, r_{1*}) = \chi(\omega)\, a(\omega), \\ \alpha(\omega)\, u(\omega,\, r_*)_{,\, r_*}\big|_{(r_*=r_{1*})} + \beta(\omega)\, v(\omega,\, r_*)_{,\, r_*}\big|_{(r_*=r_{1*})} = \chi(\omega)\, b(\omega)\,. \end{cases}$$

As functions of the still unknown constant $\chi(\omega)$ they are

$$(38)\qquad \alpha_{\mathrm{R}} = \frac{[au_{\mathrm{I}}'(r_*) - bu_{\mathrm{I}}(r_*)]\,\chi_{\mathrm{R}} - [au_{\mathrm{R}}'(r_*) - bu_{\mathrm{R}}(r_*)]\,\chi_{\mathrm{I}}}{2K(u_{\mathrm{I}},\, u_{\mathrm{R}})}\Bigg|_{r_*=r_{1*}},$$

$$(39)\qquad \beta_{\mathrm{R}} = \frac{[au_{\mathrm{I}}'(r_*) - bu_{\mathrm{I}}(r_*)]\,\chi_{\mathrm{R}} + [au_{\mathrm{R}}'(r_*) - bu_{\mathrm{R}}(r_*)]\,\chi_{\mathrm{I}}}{2K(u_{\mathrm{I}},\, u_{\mathrm{R}})}\Bigg|_{r_*=r_{1*}},$$

$$(40)\qquad \alpha_{\mathrm{I}} = \frac{[au_{\mathrm{R}}'(r_*) - bu_{\mathrm{R}}(r_*)]\,\chi_{\mathrm{R}} + [au_{\mathrm{I}}'(r_*) - bu_{\mathrm{I}}(r_*)]\,\chi_{\mathrm{I}}}{2K(u_{\mathrm{I}},\, u_{\mathrm{R}})}\Bigg|_{r_*=r_{1*}},$$

$$(41)\qquad \beta_{\mathrm{I}} = -\frac{[au_{\mathrm{R}}'(r_*) - bu_{\mathrm{R}}(r_*)]\,\chi_{\mathrm{R}} - [au_{\mathrm{I}}'(r_*) - bu_{\mathrm{I}}(r_*)]\,\chi_{\mathrm{I}}}{2K(u_{\mathrm{I}},\, u_{\mathrm{R}})}\Bigg|_{r_*=r_{1*}},$$

where I = imaginary, R = real and $K(u_{\mathrm{I}},\, u_{\mathrm{R}})$ is the quantity

$$(42)\qquad K(u_{\mathrm{I}},\, u_{\mathrm{R}}) = u_{\mathrm{I}}(r_*,\, \omega)\big|_{r_*=r_{1*}} u_{\mathrm{R}}(r_*,\, \omega) - u_{\mathrm{R}}(r_*,\, \omega)\big|_{r_*=r_{1*}} u_{\mathrm{I}}(r_*,\, \omega)\,.$$

Since $V_{(\text{ext})}(r) \to 0$ at radial infinity, we can choose as indipendent solutions $u(\omega, r_*)$ and $v(\omega, r_*)$ those having the following behaviour:

$$u(r_*) \to e^{i\omega r_*}, \qquad r_* \to \infty, \tag{43}$$

$$v(r_*) \to e^{-i\omega r_*}, \qquad r_* \to \infty. \tag{44}$$

The solution of the system (35) is given by

$$R_{2lm}(\omega, r_*) = \int_{+\infty}^{r_{1*}} G(\omega, r_*, y_*)\, S_{lm}(\omega, y_*)\, \mathrm{d}y_*. \tag{45}$$

Here $G(\omega, r_*, y_*)$ is the Green function associated to the problem that can be found by imposing the validity of the principle of causality. $G(\omega, r_*, y_*)$ results to be real:

$$G(\omega, r_*, y_*) = \left[-\frac{v(\omega, y_*)}{W}\, u(\omega, r_*) + \frac{u(\omega, y_*)}{W}\, v(\omega, r_*) \right] \vartheta(r_* - y_*), \tag{46}$$

where ϑ is the Heaviside function and W is the Wronskian of the system,

$$W = u(\omega, r_*)_{,r_*} v(\omega, r_*) - v(\omega, r_*)_{,r_*} u(\omega, r_*). \tag{47}$$

So the complete solution of eqs. (32) is

$$R_{lm}(r_*) = R_{2lm}(r_*) + R_{1lm}(r_*) = \tag{48}$$

$$= \alpha(\omega)\, u(r_*) + \beta(\omega)\, v(r_*) + \int_{r_\infty}^{r_{1*}} G(\omega, r_*, y_*)\, S_{lm}(y)\, \mathrm{d}y_*.$$

Since at the radial infinity the source term $S_{lm}(r)$ goes to zero, the asymptotic behaviour of $R_{2lm}(\omega, r_*)$ is

$$R_{2lm}(r_*, \omega) \sim \gamma(\omega)\, e^{i\omega r_*} + \varrho e^{-i\omega r_*} \tag{49}$$

and the two *complex* constants $\gamma(\omega)$ and $\varrho(\omega)$ have to be found by integrating eq. (35) numerically.

In conclusion, the asymptotic behaviour of the complete solution $R_{lm}(\omega, r_*)$ is

$$R(r_*, \omega) \sim [\gamma(\omega) + \alpha(\omega)]\, e^{i\omega r_*} + [\varrho(\omega) + \beta(\omega)]\, e^{-i\omega r_*}. \tag{50}$$

By requiring that only pure outgoing waves can emerge we have the condition

$$\gamma(\omega) + \alpha(\omega) = 0. \tag{51}$$

Remembering that the constant $\alpha(\omega)$ is known up to the constant $\chi(\omega)$, (51) allows us to determine $\chi(\omega)$ and, consequently, the coefficient $\beta(\omega)$.

The asymptotic behaviour of $R_{lm}(\omega, r_*)$ at infinity

$$R(r_*, \omega) \sim [\beta(\omega) + \varrho(\omega)]\, e^{-i\omega r_*}. \tag{52}$$

is so completely determined.

Before concluding this paragraph we want to discuss briefly the capture of a spiralling particle by a Schwarzschild black hole. The equation describing the process is still the same as the one describing perturbations outside the star but with different boundary conditions: only pure outgoing waves at radial infinity ($r_* = \infty$), and only pure ingoing waves at the black-hole horizon ($r_* = -\infty$),

$$R(r_*, \omega) = A^{\text{out}}(\omega)\, e^{-i\omega r_*}, \qquad r_* \to \infty\,, \tag{53}$$

$$R(r_*, \omega) = A^{\text{in}}(\omega)\, e^{i\omega r_*}\, r_* \to -\infty\,. \tag{54}$$

Using the Green function associated with the problem the complete solution is

$$R_{lm}(\omega, r_*) = \int_{-\infty}^{+\infty} G(\omega, r_*, y_*)\, S_{lm}(\omega, y_*)\, \mathrm{d}y_*. \tag{55}$$

By choosing as independent solutions of the homogeneous equation those having the asymptotic behaviour

$$u(r_*) = \begin{cases} e^{-i\omega r_*}, & r_* \to \infty\,, \\ C^{\text{in}}(\omega)\, e^{i\omega r_*} + C^{\text{out}}(\omega)\, e^{-i\omega r_*}, & r_* \to -\infty \end{cases} \tag{56}$$

and

$$v(r_*) = \begin{cases} D^{\text{in}}(\omega)\, e^{i\omega r_*} + D^{\text{out}}(\omega)\, e^{-i\omega r_*}, & r_* \to \infty\,, \\ e^{i\omega r_*}, & r_* \to -\infty\,, \end{cases} \tag{57}$$

the Green function is

$$\begin{cases} G(\omega, r_*, y_*) = u(r_*)\, \dfrac{v(y_*)}{W}, & y_* < r_*, \\ G(\omega, r_*, y_*) = v(r_*)\, \dfrac{u(y_*)}{W}, & y_* > r_*. \end{cases} \tag{58}$$

The asymptotic behaviour of $R_{lm}(\omega, r_*)$ is then

$$\lim_{r_* \to \infty} R(r_*, \omega) = A^{\text{out}}_{lm}(\omega)\, e^{-i\omega r_*} = \frac{e^{-i\omega r_*}}{W} \int_{-\infty}^{+\infty} v(y_*)\, S_{lm}(\omega, y_*)\, \mathrm{d}y_*. \tag{59}$$

6. – Energy spectrums

For small perturbations in a traceless, tranverse and divergenceless gauge a suitable stress-energy tensor for the gravitational radiation is

$$T_{\mu\nu} = \frac{1}{32\pi}\, \{h_{\varrho\sigma;\,\mu} h^{\varrho\sigma}{}_{;\,\nu}\}, \tag{60}$$

where the bracket denotes an avarage over several wavelenghts of the radiation. This result is also equivalent to use the Landau-Lifshitz pseudotensor in the limit of large r. Since we are dealing with Fourier transforms of the fields, averaging corresponds to taking the field amplitude times their complex conjugates

$$T_{\mu\nu} = \frac{1}{32\pi} h^*{}_{\varrho\sigma;\,\mu} h^{\varrho\sigma}{}_{;\,\nu} . \tag{61}$$

In order to apply this criterion for energy radiated we need then a traceless, tranverse and divergenceless perturbation that goes as $O(1/r)$ (the space had to be be asymptotically flat at radial infinity). The perturbation computed in the Regge-Wheeler gauge, when projected onto a Cartesian coordinate system does not have these properties. Nevertheless, a "radiative gauge" can be chosen (see ref. [2] and [23]) in which the axial perturbation has the required behaviour. In this new gauge

$$h_{lm} = -\frac{\imath}{2r^2}\sqrt{2l(l+1)(l-1)(l+2)}\, h_{2lm}(r,\omega)\, d_{lm}, \tag{62}$$

where

$$d_{lm} = -\frac{\imath r^2}{\sqrt{2l(l+1)(l-1)(l+2)}} \begin{pmatrix} (t) & (\varphi) & (r) & (\vartheta) \\ 0 & 0 & 0 & 0 \\ 0 & \sin\vartheta Z_{lm} & 0 & \sin\vartheta W_{lm} \\ 0 & 0 & 0 & 0 \\ 0 & \sin\vartheta W_{lm} & 0 & 0 - \dfrac{1}{\sin\vartheta} Z_{lm} \end{pmatrix} \tag{63}$$

is a transverse traceless tensorial harmonic and

$$Z_{lm} = 2\,\frac{\partial}{\partial\varphi}\left[\frac{\partial}{\partial\vartheta} - \cot\vartheta\right] Y_{lm},$$

$$W_{lm} = \left[\frac{\partial^2}{\partial\vartheta^2} - \cot\vartheta\,\frac{\partial}{\partial\vartheta} - \frac{1}{\sin^2\vartheta}\,\frac{\partial^2}{\partial\varphi^2}\right] Y_{lm}.$$

The relation between the two scalar functions h_{2lm} and R_{lm} is

$$h_{2lm} = \frac{2}{\omega^2}\left[\frac{\mathrm{d}(rR_{lm})}{\mathrm{d}r_*}\right]. \tag{64}$$

Consequently, the behaviour of h_{2lm} at infinity is

$$\lim_{r_*\to\infty} h_{2lm}(\omega, r_*) = \frac{2[\beta(\omega) + \varrho(\omega)]\, r\, e^{-\imath\omega r_*}}{\imath\omega} \tag{65}$$

and the emitted energy per unit frequency is

$$T_{00} = \frac{dE}{d\omega} = \sum_{l,m} \frac{l(l-1)(l+1)(l+2)}{32\pi} \left|\beta(\omega) + \varrho(\omega)\right|^2. \tag{66}$$

7. – Numerical results and conclusions

In order to compare the signals emitted when a particle falls spiralling onto stars or black holes, we have integrated eqs. (23) and (24) for stars and eq. (24) for black holes, for $l = 2$, and we have imposed that the particle starts its motion at radial infinity with the same initial conditions, *i.e.* in such a way that $T(r_* = -\infty) = 0$ and $\phi(r_* = -\infty) = 0$

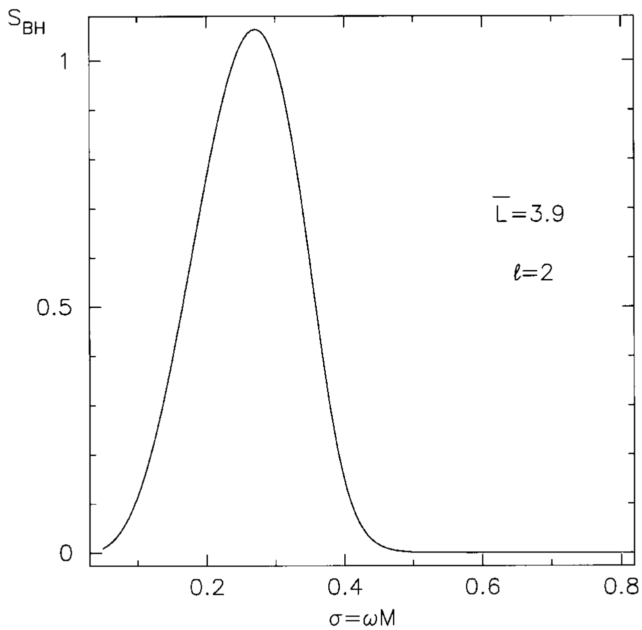

Fig. 1. – The $l = 2$ energy spectrum S_{BH} of a Schwarzschild black hole plotted as a function of $\sigma = \omega M$. $\bar{L} = 3.9$.

TABLE I – *The complex characteristic eigenfrequencies $\sigma_0 + i\sigma_i$ of a Schwarzschild black hole for $l = 2$. The imaginary part is the inverse of the damping time.*

n	$\sigma_0 = \omega_0 M$	$\sigma_i = \frac{M}{\tau}$
0	0.3737	0.089
1	0.3467	0.274
2	0.3011	0.478
3	0.2515	0.705
4	0.2075	0.947
5	0.1693	1.196
6	0.1333	1.448

when the particle reaches the horizon of the black hole. The angular momentum $\bar{L} = \tilde{L}/M$ of the particle has been set equal to 3.9. The models of stars we have considered are labelled by the $\frac{r_1}{M}$ ratio, and they will be indicated, respectively, as model I $(\frac{r_1}{M} = 2.3)$, and model II $(\frac{r_1}{M} = 2.4)$. It is known ([16]) that ultra-compact stars with a radius smaller tha $3M$ can oscillate into slowly damped axial modes. The frequency of these "trapped" modes increases with the order n, while the damping time decreases. Furthermore, there exists another set of strongly damped quasi-normal modes, the w-modes, ([17]) whose frequencies again increase with n and decreasing damping times. The question is whether these modes can be excited and how much radiation is associated to them. A qualitative answer can be given by a morphological analysis of fig. 1-6. In fig. 1 the energy spectrum emitted by a black hole is plotted as a function of the normalized frequency $\sigma = \omega M$, where ω is the frequency expressed in cm^{-1}. The spectrum has one peak at $\sigma_0 = 0.27$. In the region 0.1–0.4 where the spectrum is considerably different from zero there are seven eigenfrequencies, listed in table I (see ref. [24]). The situation is different in the case of stars. In fig. 2, 3 and 4 we plot the energy spectrum as a function of the normalized frequency $\sigma = \frac{\omega}{\sqrt{\varepsilon}}$, for model I, and in fig. 5 and 6 for model II. Figure 2 shows a large peak at the frequency of the trapped mode

$$\sigma = \omega/\sqrt{\varepsilon} = 0.4735 + \imath\, 0.26 \times 10^{-4}$$

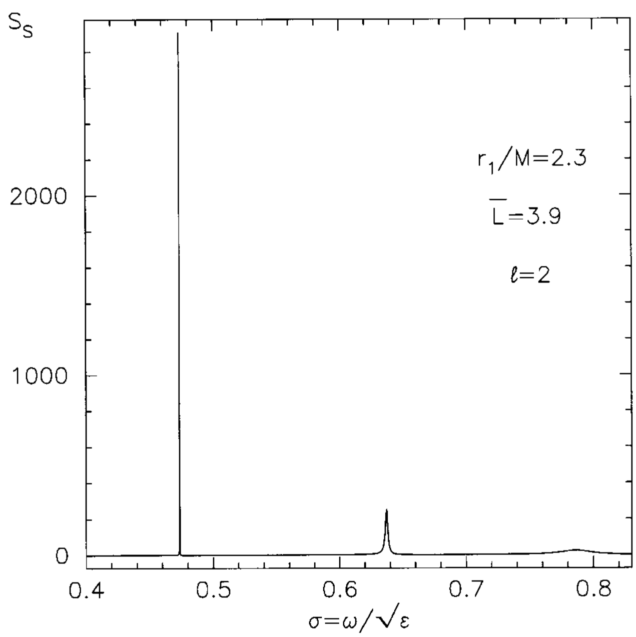

Fig. 2. – The $l = 2$ energy spectrum S_S of a homogeneous star with $r_1/M_S = 2.3$, plotted as a function of $\sigma = \omega/\varepsilon^{1/2}$. $\bar{L} = 3.9$.

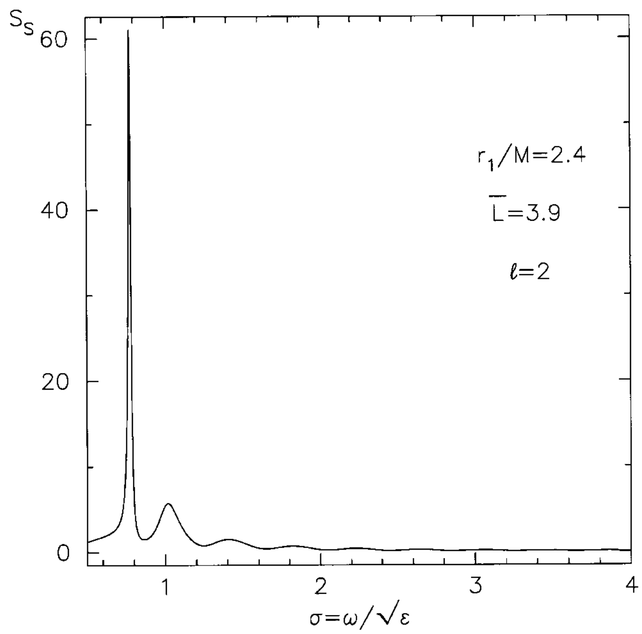

Fig. 3. – The same spectrum of fig. 2 is plotted in a range of frequency $0.75 < \sigma < 2$.

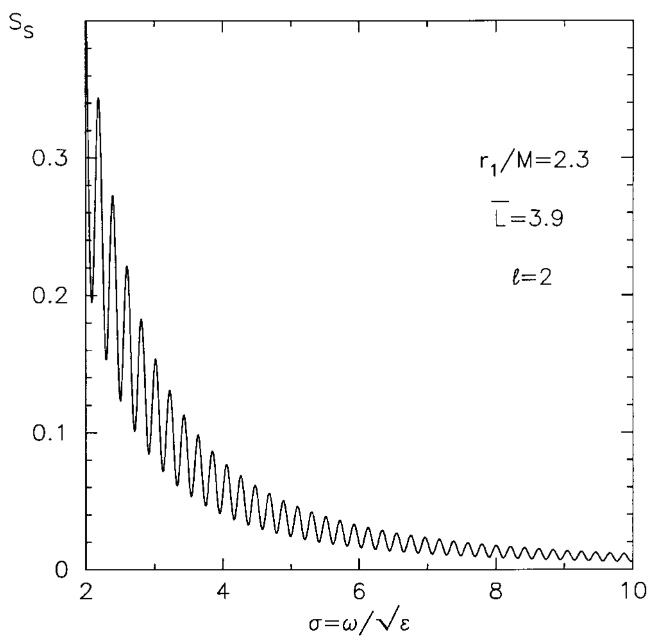

Fig. 4. – The same spectrum of fig. 2 and 3 is plotted in a range of frequency $2 < \sigma < 10$ to show the excitation of the subsequent *w*-modes.

and two smaller peaks at the other two trapped eigenfrequencies, respectively,

$$\sigma = 0.6372 + \imath\, 0.12 \times 10^{-2}, \qquad \text{and} \qquad \sigma = 0.7869 + \imath\, 0.13 \times 10^{-1}.$$

The third peak, almost invisible in fig. 2, is clearly shown in fig. 3 where the same spectrum is plotted in the region $0.75 < \sigma < 2$. From fig. 4, where the spectrum is plotted in the region $2 < \sigma < 10$, it clearly emerges that the w-modes are excited.

The second model of star (less compact than the first) has only one trapped mode for

$$\sigma = 0.7756 + \imath\, 0.92 \times 10^{-2},$$

corresponding to the first peak in fig. 5. The excitation of the w-modes is shown by the series of smaller peaks at higher frequency. Figure 6 shows the part of the spectrum for $\sigma > 1$. All the spectra we have shown are the total $l = 2$ spectra, *i.e.* the contributions of different m have been added. The results obtained with different values of the angular momentum of the infalling particle are qualitatively similar to them, only the amplitude of the signal decreases with $\bar{L}$.

A quantitative comparision is not possible because our study suffers from the assumption that we stop our analysis at the istant in which the particle reaches the surface of the star. This "cut" brings to an overstimation of the energy emitted at high frequency. However, this simple analysis does show that both classes of quasi-normal modes related to the axial perturbations of a very compact star are excited when the particle falls onto it. Our calculations also show that the total emitted energy increases with the "compactness" of the star. But the most relevant aspect of the analysis is that,

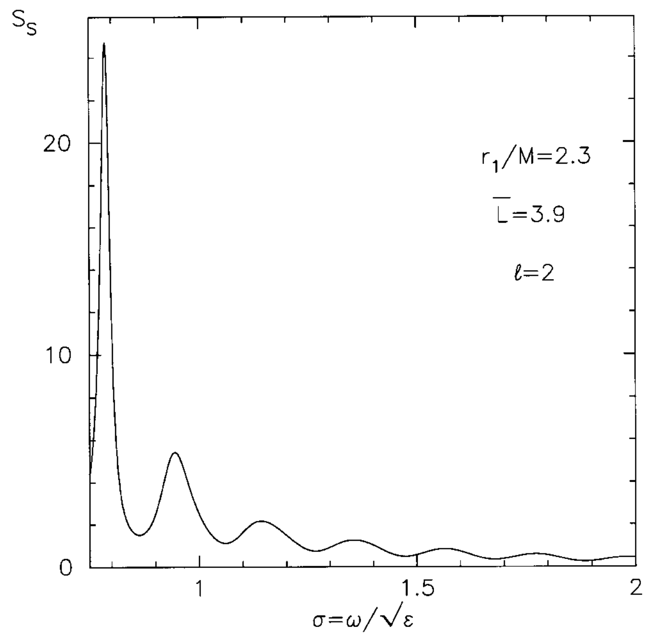

Fig. 5. – The $l = 2$ energy spectrum S_S of a homogeneous star with $r_1/M_S = 2.4$, plotted as a function of $\sigma = \omega/\varepsilon^{1/2}$. $\bar{L} = 3.9$.

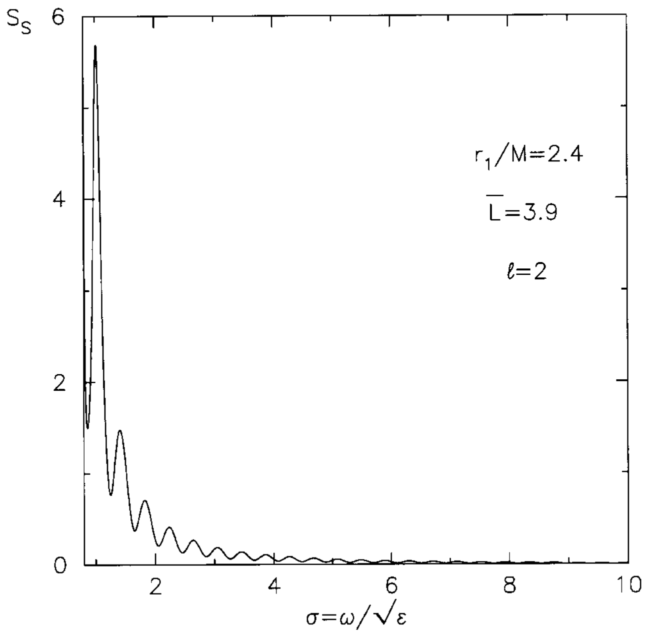

Fig. 6. – The same spectrum of fig. 5 is plotted in a range of frequency $1.4 < \sigma < 10$ to show the excitation of the *w*-modes.

due to the very different shapes of spectrums for stars and black holes, it would be possible, once the gravitational waves will be revealed, to go back up to the emitting source, *i.e.* it would be one of the (indirect) proof of the existence of black holes. We would like to stress that in this paper our aim was to give an indication of differences between the behaviour of stars and black holes in a very ideal situation. In order to obtain estimates of the radiation coming from neutron stars, more realistic equations of state should be considered as well as more exact analysis of emission due to particles scattered by or orbiting around stars.

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