

Finsler geometry in classical mechanics and in Bianchi cosmological models (*)

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Summary. — A gauge-invariant approach to study the dynamical behaviour of general Lagrangian systems is illustrated. The method is able to manage systems whose potential depends both on coordinates and velocities (possibly, on time), using a geometrical description. The manifold in which the dynamical systems live is a Finslerian space in which the conformal factor is a positively homogeneous function of first degree in the velocities (the homogeneous Lagrangian of the system). This method is a generalization of geometrodynamical approaches based on Riemannian manifolds, since it allows the study of a wider class of dynamical systems. Moreover, it is well suited to treat conservative systems with few degrees of freedom and peculiar dynamical systems whose Lagrangian is not "standard", such as the one describing the so-called *Mixmaster Universe*. We present the method and apply it to some cases of interest: 1) systems with N degrees of freedom described by conservative potentials, 2) Bianchi IX Cosmological Models (*Mixmaster Universe*), 3) the restricted three-body problem. The second example is particularly enlightening as the introduction of Finsler geometry overcomes the critical problems which cause the Jacobi (Riemannian) metric to fail.

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1. – Introduction

In the last years there has been a deep reconsideration of the very meaning of chaos, and the research has been developed towards methods (*e.g.* spectra of *stretching numbers* [1], geometrical methods [2-4]), which generalize the usual tools used to characterize the instability of trajectories, *i.e.* the Lyapunov exponents. However, for some classes of dynamical systems, other kinds of problems have been raised. For example, the usual geometrical approaches so far used (Jacobi and Eisenhart metrics) are not able to describe all the possible dynamical systems (*e.g.* non-conservative ones, systems with velocity (and/or time) dependent potentials, Lagrangians with unusual kinetic part). The aim of this paper is to present an extension of the geometrical methods capable of overcoming the limitations of standard approaches. In order to introduce such a generalization, we briefly review the fundamental steps of the geometrization of dynamics, referring the reader to the bibliography for the details [5]. In this way, we arrive at introducing the geometrization of mechanics via Finsler geometry. The Finsler geometrodynamical approach is particularly useful also because it gives a gauge-invariant characterization of chaos, which is always important, but becomes essential for General Relativistic dynamical systems.

In these last years many works have been devoted to understand the dynamical nature of the Bianchi IX Cosmological Models [6,7], but the results are controversial (see, *e.g.*, [1, 8-11]), due to the need of a gauge-invariant description and to the peculiar nature of its Lagrangian. In order to solve the first problem mentioned above, some authors ([12] and references therein), adopted the *usual* geometrical description (*i.e.* Jacobi metric), but even this approach has not been able to give a definitive answer [13], because the second problem has not been taken into account. Indeed, this dynamical system fully profits from the *Finsler geometrodynamics* as we will discuss briefly in the following sections, in which the proposed solution is also discussed, while the results are presented and interpreted in detail elsewhere [14, 15].

2. – Finsler manifold for a Lagrangian system

As is well known, Jacobi, in the celebrated lectures he gave at Königsberg in the winter 1842-43 and which were published by Clebsch in 1866, introduced a geometrization of Maupertuis's least-action principle. The aim of Jacobi was that of eliminating the time from the variational principle, so that to avoid any finalistic (or "teleological", if we want to make use of a philosophical jargon) interpretation of the principle itself. The result (of interest today) was that of stating a correspondence between the natural trajectories of the dynamical system under study and the geodesics of a Riemannian manifold with metric

$$(1) \quad ds_j^2 = 2(h - V(\mathbf{q})) ds^2,$$

being $ds^2 = \gamma_{ab} dq^a dq^b = 2\mathcal{S} dt^2 = 2(h - V(\mathbf{q})) dt^2$, where \mathcal{S} and $V(\mathbf{q})$ are the kinetic and potential energies, respectively. As regards the subject we are going to treat, two further fundamental contributions are due to Synge [4] and Eisenhart [16]. Synge gave an exhaustive treatment of classical mechanics from the geometrical point of view, by developing Jacobi's formulation and setting in a clear and rigorous framework those

which will be the fundamental tools for the study of the instability of Hamiltonian systems. On the other hand, Eisenhart introduced a metric on a manifold with $N + 2$ dimensions; starting from a dynamical system with an N -dimensional configuration space, he added a coordinate $q^0 = q^0(t)$ and a further coordinate linked to the Hamiltonian action of the system itself:

$$(2) \quad ds_E^2 = \gamma_{ab} dq^a dq^b - 2 V(\mathbf{q})(dq^0)^2 + 2 dq^0 dq^{N+1},$$

from which

$$(3) \quad ds_E^2 = 2 \left(\mathcal{L} + \frac{dq^0}{dt} \frac{dq^{N+1}}{dt} \right) dt^2,$$

\mathcal{L} being the Lagrangian of the system. To have an affine parametrization, it is necessary that the term in the round brackets is constant, that is

$$(4) \quad ds_E^2 = 2 A^2 dt^2 \Rightarrow \frac{dq^{N+1}}{dt} = A^2 - \mathcal{L}$$

(if, for the sake of simplicity, $q^0 = t$). If we put $B = q^{N+1}(0)$, we obtain

$$(5) \quad q^{N+1}(t) = A^2 t + B - \int_0^t \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) dt.$$

The geodesics of the metric (2), projected on the configuration space \mathcal{M} , coincide with the natural trajectories of the dynamical system. In any case, the Eisenhart geometry, which concerns a space $\mathcal{M} \times \mathbf{R}^2$ (space of events plus a further coordinate), also applies only to holonomic conservative systems as Jacobi's geometry does.

Let us come, finally, to Finsler's geometry. It originates (1918) as a generalization of Riemannian geometry, since it considers metrics which depend not only on the position but also on the direction and thus on the velocity (speaking in mechanical terms). Finsler's geometry was developed by Berwald (1926), Eisenhart (1927), Knebelman (1929), Cartan (1934) and Rund (1959) [17]. Let us see what novelty the introduction of the Finsler geometry has brought into the geometrization of mechanics (for details see [5]). First of all, we have to move on from the configuration space to the space of events and to introduce homogeneous variables. The dynamical system results to be characterized by the *extended Lagrangian*

$$(6) \quad \Lambda(x^\alpha, x'^\alpha) = \mathcal{L}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \frac{dt}{dw},$$

where $w = w(t)$ is a new parameter (a monotonic function of t) and

$$(7) \quad x^\alpha \equiv (q^0 = t, q^1, q^2, \dots, q^N), \quad x'^\alpha = \frac{dx^\alpha}{dw}, \quad \alpha = 0, 1, 2, \dots, N.$$

The extended Lagrangian Λ is a homogeneous function of first degree of the x'^α so that it must satisfy the following conditions:

$$(8) \quad \text{A)} \quad \Lambda(x^\alpha, kx'^\alpha) = k\Lambda(x^\alpha, x'^\alpha), \quad k > 0,$$

i.e. $\Lambda(x^\alpha, x'^\alpha)$ is positively homogeneous of degree 1 in the x'^α :

$$(9) \quad B) \quad \Lambda(x^\alpha, x'^\alpha) \neq 0 \quad \text{if all } x'^\alpha \neq 0,$$

$$(10) \quad C) \quad \frac{\partial^2 \Lambda^2(x^\alpha, x'^\alpha)}{\partial x'^\alpha \partial x'^\beta} \xi^\alpha \xi^\beta > 0, \quad \forall \xi^\alpha \neq \lambda x'^\alpha.$$

The related Euler-Lagrange equations are

$$(11) \quad \frac{d}{dw} \left(\frac{\partial \Lambda}{\partial x'^\alpha} \right) - \frac{\partial \Lambda}{\partial x^\alpha} = 0;$$

they correspond to the “old” equations

$$(12) \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) - \frac{\partial \mathcal{L}}{\partial q^i} = 0, \quad i = 1, 2, \dots, N$$

plus an identity (which comes from the first-degree homogeneity of Λ in the x'^α). Squaring the extended Lagrangian, we get a homogeneous function of second degree $F = \Lambda^2$ and we can define a Finsler metric with

$$(13) \quad ds_F^2 = \Lambda^2(x^\gamma, x'^\gamma) = g_{\alpha\beta}(x^\gamma, x'^\gamma) dx^\alpha dx^\beta, \quad g_{\alpha\beta}(x, x') = \frac{1}{2} \frac{\partial^2 \Lambda^2(x, x')}{\partial x'^\alpha \partial x'^\beta},$$

which reduces to the Riemannian metric if $\Lambda^2(x^\gamma, dx^\gamma) = g_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta$.

The geodesic equations written in the conformal parameter are

$$(14) \quad \frac{d^2 x^\alpha}{ds^2} + \gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0,$$

where (see [17], chapt. I, II)

$$(15) \quad \gamma^\alpha_{\beta\gamma} = \gamma^\alpha_{\beta\gamma}(x, x') = \frac{1}{2} g^{\alpha\delta} \left(\frac{\partial g_{\delta\beta}}{\partial x'^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial x'^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x'^\delta} \right).$$

Equations (14) have the same form of the geodesic equations in Riemannian geometry, but now the Christoffel symbols also depend on the derivatives of the x^α (*i.e.* on the velocities) with respect to w (or s). It must also be said that the Christoffel symbols occurring in the geodesic equations are different from those occurring in the covariant derivatives. Moreover, there is the possibility of defining different covariant derivatives (Berwald, Cartan, Rund [17]), but all these different forms coincide along a geodesic.

Coming back to mechanics, the geodesic equations corresponding to the metric $g_{\alpha\beta}$, if we impose, as above, the equality of w and s (*i.e.* the affinity of the parameter w) will coincide with the equations of motion of the system in the space of events. In this way one has succeeded in geometrizing a dynamical system with a Hamiltonian depending on time and potential depending also on the velocities. Thus we have a tool presenting a wider range of applicability with respect to Jacobi's and Eisenhart's geometries. An immediate advantage is given by the *gauge invariance* of the theory, since Λ is invariant under any reparametrization of the “time” w . This is of fundamental

importance, as we shall see, in the *geometrical approach* to chaos. The extension occurring in the formal treatment of the conservation laws allows the identification of the generators of the Noetherian transformations depending on velocities with *Killing vectors* of a Finslerian manifold (see the already mentioned paper *Killing equations in classical mechanics* in this volume, p. 181). We shall see in the next section the other fundamental advantages provided by the Finsler geometry in the geometrization of a dynamical system, especially for what concerns the study of chaotic dynamics.

3. – Chaotic dynamics on a Finsler space

In the last years the geometrodynamical approach in the study of chaotic dynamics has received a good deal of attention and provoked many discussions [18, 19]. Here we want to deal with dynamical systems with few degrees of freedom (“few” here means two or three), thus avoiding any debate regarding ergodicity, mixing and so on. All the systems we are going to deal with present some peculiarities which have made questionable or even impossible studying them on the basis of a geometrization *à la* Jacobi. To introduce the subject, we briefly recall the essential points of the geometrodynamical approach, starting from Jacobi geometry and then passing to its formulation in Finsler geometry. As we have recalled, the Maupertuis principle is used to reduce the equations of motion for a (conservative) dynamical system to the equations

$$(16) \quad \frac{\nabla U^i}{ds} = \frac{d^2 q^i}{ds^2} + \Gamma_{kl}^i \frac{dq^k}{ds} \frac{dq^l}{ds} = 0$$

for the geodesic flow $u^i \equiv dq^i/ds$ on the (conformal) Riemannian manifold $M: \{[q^i]: V(\mathbf{q}) \leq h\}$ equipped with the Jacobi metric (1). In (16), Γ_{kl}^i are the usual Christoffel symbols of Riemannian geometry. In order to investigate the *stability* of the flow on M , the so-called *stability tensor* [2, 18]

$$(17) \quad \mathcal{R}^a{}_c \stackrel{\text{def}}{=} R^a{}_{bcd} u^b u^d$$

has been introduced, whose (invariant) trace is the *Ricci curvature along the flow*

$$(18) \quad \text{tr } \mathcal{R} \equiv \mathcal{R}^a{}_c = R_{bd} u^b u^d \equiv \text{Ric}(u).$$

The growth of a perturbation δq^a to a given geodesic is described by the Jacobi–Levi-Civita equation of geodesic deviation

$$(19) \quad \frac{\nabla}{ds} \left(\frac{\nabla \delta q^a}{ds} \right) = - \mathcal{R}^a{}_c \delta q^c.$$

In fact (see [2, 18]), the evolution of a perturbed geodesic depends on the eigenvalues of the stability tensor, which are nothing else than the sectional curvatures in the $n - 1$ layers containing the tangent vector to the actual flow. The Jacobi-geometry approach, however, has severe limitations. First of all, it can only be applied to holonomic conservative systems and with potentials not depending on velocity. Moreover, the time t is not an affine parameter for the geodesics, in the sense that dt^2 is not proportional (through a constant) to ds_t^2 , because the kinetic energy is obviously not constant. In addition, the theory is not invariant under a reparametrization and

then not “gauge invariant”: this makes questionable any application to systems in the general-relativistic context, where this invariance is required. At last, Jacobi’s metric becomes singular for $\mathcal{F} = 0$, and then in the equilibrium points and in the points where the motion is inverted. All these limitations can be overcome by introducing Finsler geometry, that is by studying the geodesic flow on a Finsler manifold. Now (14) are the relevant geodesic equations.

The information about the possible *chaotic* behaviour of the dynamical system is fully contained in the equation of the geodesic deviation, only formally analogous to the Jacobi-Levi-Civita [2, 3] one. It is given by

$$(20) \quad \frac{\delta^2 Z^\alpha}{\delta S^2} + K^\alpha_{\beta\gamma\delta}(X, X') X'^\beta X'^\delta Z^\gamma = 0,$$

where $\delta/\delta S$ is the so-called δ -differentiation and $K^\alpha_{\beta\gamma\delta}(X, X')$ is “one” of the curvature tensors which can be defined in this manifold. Its expression contains some terms involving the derivatives of the velocities in addition to the usual terms present in the Riemann tensor; it is

$$(21) \quad K^\alpha_{\beta\gamma\delta}(X, X') \stackrel{\text{def}}{=} A^\alpha_{\beta\gamma\delta} - A^\alpha_{\beta\delta\gamma} + \Gamma_{\mu\delta}^{*\alpha} \Gamma_{\beta\gamma}^{*\mu} - \Gamma_{\mu\gamma}^{*\alpha} \Gamma_{\beta\delta}^{*\mu},$$

where

$$(22) \quad A^\alpha_{\beta\gamma\delta} \stackrel{\text{def}}{=} \frac{\partial \Gamma_{\beta\gamma}^{*\alpha}}{\partial X^\delta} - \frac{\partial \Gamma_{\beta\gamma}^{*\alpha}}{\partial X'^\nu} \frac{\partial \mathcal{G}^\nu}{\partial X'^\delta},$$

$$(23) \quad 2\mathcal{G}^\alpha = \gamma^\alpha_{\beta\gamma} X'^\beta X'^\gamma = \frac{1}{2} g^{\alpha\delta} \left[\frac{\partial g_{\delta\beta}}{\partial X^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial X^\beta} - \frac{\partial g_{\beta\gamma}}{\partial X^\delta} \right] X'^\beta X'^\gamma,$$

$$(24) \quad \Gamma_{\beta\gamma}^{*\alpha} \stackrel{\text{def}}{=} \gamma^\alpha_{\beta\gamma} - g^{\mu\alpha} \left(C_{\gamma\mu\nu} \frac{\partial G^\nu}{\partial X'^\beta} + C_{\beta\mu\nu} \frac{\partial G^\nu}{\partial X'^\gamma} - C_{\beta\gamma\nu} \frac{\partial G^\nu}{\partial X'^\mu} \right),$$

$$(25) \quad \mathcal{G}_{\alpha\beta\gamma} = \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial X'^\gamma}, \quad \mathcal{G}^\delta_\alpha = \frac{\partial \mathcal{G}^\delta}{\partial X'^\alpha}.$$

As in Riemannian geometry, we define a *stability tensor* [2, 20]:

$$(26) \quad H^\alpha_\delta \stackrel{\text{def}}{=} K^\alpha_{\beta\gamma\delta} X'^\beta X'^\gamma$$

which contains all the information about the dynamical behaviour of the system.

In the following we derive the main equations for three cases: 1) conservative potentials for N degrees of freedom; 2) the Lagrangian for the Cosmological Bianchi type-IX models; 3) the restricted three-body problem. For these problems (in particular in the first example, when N is two or three), we discuss the advantages that the Finsler metric has with respect to the Jacobi one.

4. – The space of events as a Finsler space

We consider now a completely general system with monogenic forces, in which the kinetic energy can contain also terms linear in the velocities and the potential can be

also explicitly time-dependent:

$$(27) \quad \mathcal{L}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) = \mathcal{F}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) - V(t, \mathbf{q}).$$

According to the prescription introduced in sect. 2, the homogeneous Lagrangian of eq. (6) is therefore

$$(28) \quad \Lambda(x, x') = \left[\mathcal{F}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) - V(t, \mathbf{q}) \right] \frac{dt}{dW}.$$

In the particular case of a natural conservative system, Λ would be given by

$$(29) \quad \Lambda(x, x') = \frac{1}{2} \gamma_{ab}(x) x'^a x'^b (x'^0)^{-1} - V(x) x'^0,$$

where

$$(30) \quad x'^0 = \frac{dx^0}{dW} = \frac{dt}{dW}, \quad x'^a = \frac{dx^a}{dW}.$$

Having defined a metric through the position of eq. (13), in the case of a natural conservative system, its components have non-trivial expressions in terms of the “old” quantities:

$$(31) \quad \begin{cases} g_{00} = \frac{1}{2} \frac{\partial^2 \Lambda^2}{\partial^2 (x'^0)^2} = 3\mathcal{F}^2 + V^2, \\ g_{0a} = \frac{1}{2} \frac{\partial^2 \Lambda^2}{\partial x'^0 \partial x'^a} = -2\mathcal{F} \gamma_{ab} \frac{x'^b}{x'^0}, \\ g_{ab} = \gamma_{ac} \gamma_{bd} x'^c x'^d (x'^0)^{-2} + \gamma_{ab} (\mathcal{F} - V). \end{cases}$$

The Lagrange equations of the system with Lagrangian (28) will coincide with the geodesic equations of the Finsler space with the metric (13).

4.1. Conservative systems. – As has been shown in [2, 18, 19, 3], the evolution of a perturbation to a geodesic in the Jacobi metric, for a system with N degrees of freedom, is governed by the *Ricci curvature along the flow*, which is the trace of the stability tensor, given by

$$(32) \quad \text{tr } H_J = \frac{1}{2\mathcal{F}^2} \left\{ \Delta V + \frac{(\nabla V)^2}{\mathcal{F}} + (N-2) \left[\frac{1}{2} \left(\frac{dV}{ds_J} \right)^2 + \mathcal{F} \frac{d^2 V}{ds_J^2} \right] \right\},$$

where $\mathcal{F} \equiv h - V$ is the kinetic energy (h and V being the total and the potential energies, respectively). In analogy, a simple indicator of stability for a Finsler geodesic is the trace of H_F^j :

$$(33) \quad \text{tr } H_F = t'^2 \Delta V + N t' \frac{d^2 V}{ds^2} + N t'^2 \left(\frac{dV}{ds} \right)^2,$$

where $t' = 1/\mathcal{L}$ is the inverse of the Lagrangian. The difference between $\text{tr } H_F$ and its analogue in the Jacobi metric consists in the absence of the gradient and, more

important, in the presence of the Lagrangian instead of the kinetic energy. This fact has very important implications because it allows to overcome the singularity in the conformal factor, unavoidable in the Jacobi metric. Indeed in the definition of the Lagrangian there is a gauge freedom which permits to make it sign-definite (*i.e.* \mathcal{L} never vanishes). This is clearly more important in the case of few degrees of freedom, or when the system is near to the integrability, as in both cases the chance that the kinetic energy vanishes is not negligible.

4.2. Finsler's manifold for Bianchi IX models. – The wider applicability of the Finsler geometrodynamics is strikingly evident in the case of the Bianchi IX models [14] for which the vanishing Hamiltonian is expressed by [21]

$$(34) \quad \frac{\dot{\beta}_+^2}{2} + \frac{\dot{\beta}_-^2}{2} - \frac{\dot{\alpha}^2}{2} + V(\alpha, \beta_+, \beta_-) = 0,$$

where β_+ , β_- and α are functions of the scale factors a, b, c of the *Mixmaster Universe*, the dot means differentiation with respect to τ ($d\tau = dt/abc$, t is the proper time) and V is the potential of the system. We can construct the following homogeneous Lagrangian [22, 14, 15]:

$$(35) \quad \Lambda = \frac{1}{2\tau'} (\alpha'^2 - \beta_+'^2 - \beta_-'^2) - U(\alpha, \beta_+, \beta_-) \tau',$$

where

$$(36) \quad U(\alpha, \beta_+, \beta_-) = -V(\alpha, \beta_+, \beta_-) + C,$$

in which the gauge is fixed by the choice of the (negative) constant C added in order to make the homogeneous Lagrangian positively definite.

For what concerns the geodesic deviation equation, we calculate the trace of the *stability tensor*:

$$(37) \quad H_F = \text{tr } H^j_j = \tau'^2 \Delta V - 3\tau' \frac{d^2 V}{ds^2} + 3\tau'^2 \left(\frac{dV}{ds} \right)^2,$$

where we have defined

$$(38) \quad \Delta V \stackrel{\text{def}}{=} \frac{\partial^2 V}{\partial \beta_+^2} + \frac{\partial^2 V}{\partial \beta_-^2} - \frac{\partial^2 V}{\partial \alpha^2}.$$

In this expression there is a positive term, and two terms whose sign is not clearly definite. This seems to suggest that also in this particular dynamical system, the origin of the dynamical instability is not (or not only) related to the negativity of (some) curvature, but rather to the violent fluctuations of the geometrical quantities (see, *e.g.*, [2, 3, 19]). If this is the case, nothing can be said about the relation between the instability time of trajectories and relaxation properties of the system. Let us now consider eq. (34), and let us introduce it in the expression of the trace of the stability tensor (37). We see that

$$(39) \quad \tau' = \frac{1}{2U - C},$$

with $C < 0$, so that $\tau' > 0$. In order to compare the Finsler metric with the Jacobi one, we calculated also the trace of the *stability tensor* in the Jacobi metric for this dynamical system [13, 2, 3]:

$$(40) \quad H_J = \frac{1}{2W^2} \left[\Delta V + \frac{(\nabla V)^2}{W} + \frac{1}{2} \left(\frac{dV}{ds} \right)^2 + W \frac{d^2 V}{ds^2} \right],$$

where $W = \dot{\beta}_+^2/2 + \dot{\beta}_-^2/2 - \dot{\alpha}^2/2$. Now also the term

$$(41) \quad (\nabla U)^2 \stackrel{\text{def}}{=} \left(\frac{\partial V}{\partial \beta_+} \right)^2 + \left(\frac{\partial V}{\partial \beta_-} \right)^2 - \left(\frac{\partial V}{\partial \alpha} \right)^2$$

is present. It is clear why the Jacobi metric cannot work: the trace diverges when the “kinetic energy” W (or the potential V) vanishes and, along each trajectory, this happens infinitely many times (passing from one *Kasner’s epoch* to another) going towards the singularity. On the contrary, this does not occur in the Finsler manifold we introduced. We can better understand this by considering the line element of the two metrics, the Jacobi one and the Finsler one:

$$(42) \quad ds_J = -\sqrt{2} V d\tau,$$

$$(43) \quad ds_F = (2V - c) d\tau.$$

So, while $ds_J = 0$ if $\mathcal{S} = V = 0$ (this cannot be avoided adding a constant), this does not happen for ds_F because in this case the conformal factor is the Lagrangian of the system to which we can add a constant without changing the equations of motion. Numerical simulation are running, in order to estimate averages and fluctuations of geometrical quantities related to the Finslerian transcription of BIX dynamics. The results will be presented elsewhere[15].

43. *The restricted three-body problem.* – So far, chaotic motions in the restricted three-body problem have been studied via numerical explorations by means of the two methods of the surface of section and the Lyapunov exponents (see, e.g., [23-26]). Owing to the already mentioned reason of the presence in the Lagrangian of the terms linear in the velocity, a Jacobi geometrization is not possible: only through the introduction of a Finsler manifold it is possible to geometrize the three-body restricted problem.

Assuming the plane of motion as the (x, y) -plane, the Lagrangian in the rotating system of reference (choosing the measure units so to have $m_1 + m_2 = 1$, $r = 1$, $G = 1$) is given by

$$(44) \quad \mathcal{L} = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) + (\dot{x}y - \dot{y}x) - V(x, y),$$

with $\dot{x} = dx/dt$, $\dot{y} = dy/dt$ and

$$(45) \quad V(x, y) = -\frac{1}{2} (x^2 + y^2) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2},$$

where $r = x_2 - x_1 = r_1 + r_2$, $\mu = \text{dimensionless } m_2 < 1/2$, and the equations of motion are given by

$$(46) \quad \ddot{x} - 2\dot{y} = -\frac{\partial V}{\partial x}, \quad \ddot{y} - 2\dot{x} = -\frac{\partial V}{\partial y}.$$

The extended Lagrangian results to be

$$(47) \quad \Lambda = \frac{1}{2t'}(x'^2 + y'^2) + (xy' - yx') - V(x, y) t',$$

where the prime means d/dw . For the metric tensor, we obtain

$$(48) \quad \left\{ \begin{array}{l} g_{tt} = \frac{3\mathcal{F}^2}{t'^2} + V^2 + \frac{2\mathcal{F}}{t'^3}(-x'y + y'x), \\ g_{xx} = \frac{x'^2}{t'^2} + \frac{\Lambda}{t'} - \frac{2-x'y}{t'} + y^2, \\ g_{yy} = \frac{y'^2}{t'^2} + \frac{\Lambda}{t'} + \frac{2-y'x}{t'} + x^2, \\ g_{tx} = -\left(\frac{\mathcal{F}}{t'^2} + V\right)\left(\frac{x'}{t'} - y\right) - \frac{x'}{t'^2}\Lambda, \\ g_{ty} = -\left(\frac{\mathcal{F}}{t'^2} + V\right)\left(\frac{y'}{t'} + x\right) - \frac{y'}{t'^2}\Lambda, \\ g_{xy} = \left(\frac{x'}{t'} - y\right)\left(\frac{y'}{t'} + x\right) \end{array} \right.$$

and

$$(49) \quad d(g_{\alpha\beta}) = \frac{\Lambda^4}{t'^4}.$$

The trace of the stability tensor results to be

$$(50) \quad \text{tr } H^{\alpha}_{\beta} = t' \Delta V + 2t' \left[\frac{dA}{ds} + t' A^2 \right] + 2t'^2,$$

with

$$(51) \quad A = \frac{dV}{ds} + xx' + yy' - \frac{t'}{2}(V_x y - V_y x),$$

where

$$(52) \quad V_x = \frac{\partial V}{\partial x}, \quad \Delta V = V_{xx} + V_{yy}$$

and, in (51) and (52) we have made $w = s$. As for the preceding case, numerical calculations are running and the results have been presented in the preceding article of this volume [5].

5. – Conclusions

We have shown how a dynamical system is described in a Finsler manifold giving some examples. The introduction of a Finsler manifold has been crucial in overcoming the singularities which occur in the Jacobi metric when the kinetic energy vanishes. The frequency of such an occurrence is not negligible when these singularities are inside the manifold. The most relevant result we reach is that, by means of this kind of approach, it is possible to give a gauge-invariant description of those systems whose potential depends also on velocities (*e.g.*, the restricted three-body problem [27]) and, moreover, whose Lagrangian is not a positively definite quadratic form in the velocities. This will be presented elsewhere [27, 20].

As we have seen, given a dynamical system subjected to monogenic forces and then represented in homogeneous variables by a Lagrangian function $\Lambda(x, x')$ in which, in general, also the time coordinate $t = x^0$ can appear, it is always possible “to geometrize” it completely, *i.e.* to incorporate the generalized work function into the metric. In the examples we have considered, the motion of a particle in given potentials, one obtained the motion of a free particle in a curved space (Finsler manifold) whose metric tensor was depending also on the direction (velocity). The trajectories allowed in the actual motion thus became the geodesics of this curved space. It is known that it is possible to derive from a Lagrangian also particular dissipative forces depending only on the velocities and, in case, on the time [28, 29] by means of a suitable transformation of the independent variable.

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