Killing equations in classical mechanics (*)

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Summary. — The relation between Noether theorem and the generalized symmetries of Riemannian and Finsler metrics is explored to set in a more general framework the search of conserved quantities in classical mechanics. The *Killing-tensor* equations are written in the Jacobi metric, which constitutes the simplest way to "geometrize" the evolution of a conservative Lagrangian system: the relation between the additional invariant in the physical gauge and the Killing-tensor invariant on the Jacobi manifold is written in the general case and is explicited in several classical cases, for which all the Killing tensors associated to quadratic invariants at arbitrary energy are found. This approach allows to interpret the Killing tensors as the generators of generalized Noetherian transformations depending also on the velocities. After that, the *Killing-vector* equations are written in the Finsler metric, which allows a more general geometrization of dynamical problems, including non-conservative systems and Lagrangian with terms linear in the velocities. In fact, exploiting the velocity dependence of tensorial objects in the Finsler metric, Killing vectors in this case give also the invariants of higher order in the momenta.

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1. – Introduction

The problem of the relation existing between the symmetries of a given physical system and its conservation laws has intrigued the mathematical physicists for a long time and it is not even easy to establish when the first concrete results were achieved.

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Perhaps, Jacobi, in his *Vorlesungen*, was the first to derive the conservation of momentum and angular momentum from the invariance of the Lagrangian under translations and rotations. It is, however, sure that the fundamental result is embodied in the theorem stated by Emmy Noether in 1918 [1]. Of this theorem many proofs and generalizations have been given (see, *e.g.*, [2]), using several frameworks. The aim of the present paper is to put in evidence the link between two, seemingly different, ways of proving the existence of constants of the motion for a given system: the Noether theorem just referred to and the method based on the existence of Killing vectors and, more generally, tensors [3] in a given Riemannian manifold. The equivalence of the two approaches to the conservation laws for a dynamical system will result completely general in the framework provided by the homogeneous Lagrangian variables and the consequent "geometrization" in a Finsler space. Historically, the two approaches have had, we can say, parallel lives and this has been also due to having been born in different fields of Mathematical Physics: the theory of General Relativity and Classical Mechanics. Separately, they have received quite a lot of attention and as many have been the applications of the Noether theorem in the classical mechanics and field theory (also quantum field theory) so have been the applications in the theory of General Relativity and relativistic cosmologies of the Killing equations.

The relation between the Killing vectors and tensors admitted by the four-dimensional manifold and the constants of the motion of the geodesics is largely known to people working in the General Relativity theory. But only in the last decades [4] it has been pointed out that the relations that have to be satisfied so that a transformation be Noetherian are given by (generalized) Killing equations. It matters, therefore, to study the relation between the Killing tensors of a suitable Riemannian manifold (and after them the vectors of a Finsler manifold) and the first integrals of a dynamical system having a given Lagrangian. In order to accomplish the unitary picture in which to merge the two different ways of tackling the problem of the existence of the constants of the motion, we have been obliged to re-expound results which are widely well known. In such a case, only the enunciations will be given, referring the reader to the original papers for the demonstrations. It has been our care to provide examples taken from the classical ones to make the application of the criteria we have obtained more immediately evident. We will limit ourselves to the consideration of discrete systems (such as, for example, a system of N material points), but the treatment has a more general validity.

The plan of the paper is as follows: in sect. **2** we give a concise exposition of the Noether theorem in the usual Lagrangian formalism, of the respective inverse theorem and of the extension of the theorem to the cases where also forces not deriving from a Lagrangian are present. We point out a special consequence of the inverse theorem which allows the reduction of Noether transformations which are purely spatial but involving a gauge function to transformations which concern also the time variable but with a vanishing gauge function. In sect. **3** we introduce the homogeneous Lagrangian variables and rewrite the Noether theorem. The treatment of sect. **2** is the basis, in sect. **4**, for the comparison with the approach which has, as starting point for the search of conserved quantities, the singling out of Killing vectors and tensors of a given Riemannian manifold (in this case the manifold with "Jacobi metric" relevant to a mechanical system with fixed total energy). Such an approach applies only to conservative systems. In sect. **5** the application to systems with two degrees of freedom will be considered and in sect. **6**, referring to this, some classical cases are reviewed. Section **7** is devoted to the systems which can be non-conservative but subjected to forces which must in any case derive from a (generalized) potential function. We show that for such systems the "geometrization" must necessarily take place in a Finsler space in which the coordinates are given by the homogeneous Lagrangian coordinates introduced in sect. **3**. It is therefore the formulation of the Noether theorem in the space of events which will be discussed in connection with the generalized Killing vectors in such a space. In sect. **8** the conclusions of the work are sketched.

2. – The Noether theorem in the configuration space

We give here, with the aim of having in the following the relevant formulas to refer to, the statement of the Noether theorem (limited to the case of one-parameter transformations) and an "*intuitive sketch*" of its proof; we refer, for a rigorous proof, to the papers cited above. Given a Lagrangian

$$\mathscr{L} = \mathscr{L}(t,\, \mathbf{q}(t), \dot{\mathbf{q}}(t))\,, \tag{1}$$

characterizing a system subjected to monogenic (see [5], p. 30)) forces, we shall consider infinitesimal transformations of the time and of the configurational variables of the form

$$\bar{t} = t + \tau(t,\, \mathbf{q}, \dot{\mathbf{q}})\,\varepsilon\,, \qquad \bar{q}^k = q^k + \xi^k(t,\, \mathbf{q}, \dot{\mathbf{q}})\,\varepsilon\,, \qquad k = 1, 2, \ldots, N\,, \tag{2}$$

such that the action integral be invariant up to a *gauge* term, that is, for any differentiable curve $t \to \mathbf{q}(t)$, a function f exists such that

$$\int\limits_{\bar{t}_1}^{\bar{t}_2} \mathscr{L}\left[\bar{t}, \bar{\mathbf{q}}(\bar{t}),\, \frac{\mathrm{d}}{\mathrm{d}\bar{t}}\,\bar{\mathbf{q}}(\bar{t})\right] \mathrm{d}\bar{t} = \int\limits_{t_1}^{t_2} \mathscr{L}(t,\, \mathbf{q}(t), \dot{\mathbf{q}}(t))\,\mathrm{d}t + \varepsilon \int\limits_{t_1}^{t_2} \frac{\mathrm{d}f(t,\, \mathbf{q}, \dot{\mathbf{q}})}{\mathrm{d}t}\,\mathrm{d}t + o(\varepsilon)\,, \tag{3}$$

with $[t_1, t_2] \subset [a, b]$ and $[a, b] \subset \mathbb{R}$. The parameter ε in eqs. (2) and (3) will be such that $\varepsilon \in I \subset \mathbb{R}$, where I contains the origin as interior point and the transformation of the velocities, obtained by differentiating eq. (2), is given to first order in ε by

$$\frac{\mathrm{d}\bar{q}^k(\bar{t})}{\mathrm{d}\bar{t}} = \frac{\mathrm{d}q^k}{\mathrm{d}t} + \varepsilon(\dot{\xi}^k - \dot{q}^k\dot{\tau})\,. \tag{4}$$

It is not difficult to show that the conditions for this are the equations

$$\mathscr{L}\,\frac{\partial\tau}{\partial\dot{q}^i} + \frac{\partial\mathscr{L}}{\partial\dot{q}^j}\left(\frac{\partial\xi^j}{\partial\dot{q}^i} - \frac{\partial\tau}{\partial\dot{q}^i}\,\dot{q}^j\right) = \frac{\partial f}{\partial\dot{q}^i}\,, \qquad i, j = 1, 2, \ldots, N \tag{5}$$

and

$$\frac{\partial\mathscr{L}}{\partial t}\,\tau + \frac{\partial\mathscr{L}}{\partial q^i}\,\xi^i + \frac{\partial\mathscr{L}}{\partial\dot{q}^i}\left[\frac{\partial\xi^i}{\partial t} + \frac{\partial\xi^i}{\partial q^j}\,\dot{q}^j - \dot{q}^i\left(\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial q^j}\,\dot{q}^j\right)\right] + \\ + \mathscr{L}\left(\frac{\partial\tau}{\partial t} + \frac{\partial\tau}{\partial q^i}\,\dot{q}^i\right) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^i}\,\dot{q}^i. \tag{6}$$

The system (5) and (6) is constituted by $N+1$ partial differential equations, linear in the $N+1$ unknown functions τ and ξ^i and represents the necessary and sufficient conditions in order that the action integral be invariant up to *gauge* terms under the infinitesimal transformations with generators τ and ξ^i. Equations (5) and (6) were called by Logan [4] generalized Killing equations. In fact it is easy to see that, when

$$\mathscr{L} = \frac{1}{2}\gamma_{kl}\dot{q}^k\dot{q}^l \tag{7}$$

(a system of free or holonomically constrained mass points) and one considers a transformation

$$\bar{t} = t\,, \qquad \bar{q}^k = q^k + \xi^k(q)\,\varepsilon\,, \tag{8}$$

they are replaced by

$$\gamma_{il,k}\xi^k + \gamma_{ik}\frac{\partial\xi^k}{\partial q^l} + \gamma_{kl}\frac{\partial\xi^k}{\partial q^i} = 0\,, \tag{9}$$

which are just the Killing equations (see, for instance, [6]).

Moreover, the consequence on the system of the invariance of the action integral under the transformations we have considered is that, if the Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathscr{L}}{\partial\dot{q}^i}\right) - \frac{\partial\mathscr{L}}{\partial q^i} = 0 \tag{10}$$

are satisfied, then the total time derivative vanishes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathscr{L}\tau + \frac{\partial\mathscr{L}}{\partial\dot{q}^i}(\xi^i - \dot{q}^i\tau) - f\right] = 0\,, \tag{11}$$

that is the quantity in square brackets in eq. (11) is a constant of the motion. For instance, in the case of the transformation (8) ($\tau\equiv 0$ and with also $f\equiv 0$), the constant of the motion is

$$I = \xi^i p_i\,, \tag{12}$$

being $p_i = \partial\mathscr{L}/\partial\dot{q}^i$.

The result in eq. (11) constitutes the conclusion of the Noether theorem. Conversely, owing to what we have already demonstrated, we can say that necessary and sufficient condition in order that a transformation be *Noetherian*, is that its generators satisfy the generalized Killing equations (5) and (6). For what concerns these equations, it is clear that they can be used in two ways which are, so to say, complementary. If we start with a given Lagrangian $\mathscr{L}$ and a transformation with generators τ and ξ^i, (5) and (6) enable us to check if the transformation is "Noetherian" and then, from (11), to obtain the corresponding constant of the motion. Conversely, if a transformation is given with known generators τ and ξ^i, (5) and (6) can be used to single out the set of Lagrangians enjoying the property of invariance under the given transformation. In the first case, it may also happen that one has the Lagrangian $\mathscr{L}$ and wants to determine *all* the possible Noetherian transformations. This is the case in which we are interested. Being it impossible, for obvious reasons, to determine a complete integral of (5) and (6), in this case one will be obliged to make systematic attempts starting from the simplest and obvious cases. A particular example of this will be done in the next sections.

The enunciation of the Noether theorem we have given above is limited to systems subjected to monogenic forces, that is to systems for which a work function (possibly time-dependent) exists from which all forces can be derived. However, it is possible to extend the theorem also to the case in which forces exist which cannot be derived from a Lagrangian. In this case, if we denote by Q_i the components of these forces, the equations of motion will be

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}^i}\right) - \frac{\partial \mathscr{L}}{\partial q^i} = Q_i . \tag{13}$$

It has been demonstrated (see Dyukic and Vujanovic, [4]) that, in this case, the transformation of the velocities $\dot{q}^i$ cannot be obtained as in (4) by a simple differentiation of eq. (2), but must be independently stated:

$$\frac{\mathrm{d}\bar{q}^k(\bar{t})}{\mathrm{d}\bar{t}} = \frac{\mathrm{d}q^k}{\mathrm{d}t} + \varepsilon[\dot{\xi}^k - \dot{q}^k\dot{\tau} + \Phi^k(t, \mathbf{q}, \dot{\mathbf{q}})] , \tag{14}$$

introducing new functions $\Phi^k(t, \mathbf{q}, \dot{\mathbf{q}})$. If the action integral is invariant up to gauge terms under the transformations (2) and (14) and if, moreover, the identity

$$\frac{\partial \mathscr{L}}{\partial \dot{q}^i}\Phi^i = Q_i(\dot{\xi}^k - \dot{q}^k\dot{\tau}) \tag{15}$$

holds, then there exists the first integral given by the expression in the square brackets in (11). Equations (5) remain unchanged, while (6) becomes

$$\frac{\partial \mathscr{L}}{\partial t}\tau + \frac{\partial \mathscr{L}}{\partial q^i}\xi^i + \frac{\partial \mathscr{L}}{\partial \dot{q}^i}\left[\frac{\partial \xi^i}{\partial t} + \frac{\partial \xi^i}{\partial q^j}\dot{q}^j - \dot{q}^i\left(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial q^j}\dot{q}^j\right)\right] + \mathscr{L}\left(\frac{\partial \tau}{\partial t} + \frac{\partial \tau}{\partial q^i}\dot{q}^i\right) + Q_i(\dot{\xi}^k - \dot{q}^k\dot{\tau}) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^i}\dot{q}^i . \tag{6'}$$

Therefore, the Noetherian transformation of the velocities is determined by the non-conservative forces Q_i, besides the transformations of the q^i and t.

If we now consider the first integral in (11)

$$I = \mathscr{L}\tau + \frac{\partial \mathscr{L}}{\partial \dot{q}^i}(\xi^i - \dot{q}^i\tau) - f \tag{16}$$

and differentiate it with respect to the $\dot{q}^j$ and substitute eq. (5) in the obtained expression we shall have finally

$$\xi^i = J^{ij}\frac{\partial I}{\partial \dot{q}^j} + \dot{q}^i\tau , \tag{17}$$

where

$$J^{ij}\frac{\partial^2 \mathscr{L}}{\partial \dot{q}^j \partial \dot{q}^k} = \delta^i{}_k \tag{18}$$

and the Hessian determinant is assumed not vanishing. By substituting eq. (17) in (16), we shall have for τ:

(19) $$\tau = \mathscr{L}^{-1}\left(I + f - J^{ij}\,\frac{\partial I}{\partial \dot{q}^j}\,\frac{\partial \mathscr{L}}{\partial \dot{q}^i}\right).$$

When the term quadratic in the velocities appears in the Lagrangian in the simple form $(1/2)\delta_{ij}\dot{q}^i\dot{q}^j$, it is easy to see that also J^{ij} becomes a Kronecker δ. At this point, we can enunciate the inverse Noether theorem (in the case of monogenic forces): "To any constant of the motion $I = I(\mathbf{q}, \dot{\mathbf{q}}, t)$ for the dynamical system described by the Lagrangian $\mathscr{L}$ it corresponds an infinitesimal transformation, given by (17) and (19) which leaves the action integral invariant up to gauge terms for all the solutions $t \to \mathbf{q}(t)$ of the equations of motion". Since in eq. (19) the function $f = f(\mathbf{q}, \dot{\mathbf{q}}, t)$ is quite arbitrary, it follows that also the function $\tau = \tau(\mathbf{q}, \dot{\mathbf{q}}, t)$ is arbitrary and thence eq. (17) does not define a coordinate transformation, but a family of coordinate transformations. This consequence, which *a priori* seems to introduce a troublesome arbitrariness, as a matter of fact enables us to "simplify" the transformation corresponding to a given integral. To this end, one can derive a proposition of useful application in concrete cases. Let us assume we have a first integral $I = \bar{I}(\mathbf{q}, \dot{\mathbf{q}}, t)$ and, moreover, this integral, through the Noether theorem, corresponds to the transformation defined by

(20) $$\tau = \bar{\tau} \equiv 0\,, \qquad \xi^i = \bar{\xi}^i(\mathbf{q}, \dot{\mathbf{q}}, t)\,,$$

with $f = \bar{f}(\mathbf{q}, \dot{\mathbf{q}}, t)$; then, as a consequence, an infinity of transformations like

(21) $$\xi^i = \bar{\xi}^i + \tau\dot{q}^i\,, \qquad \tau = \tau(\mathbf{q}, \dot{\mathbf{q}}, t)\,,$$

results to be defined, with $f = \bar{f} + \tau\mathscr{L}$ (τ quite an arbitrary function), corresponding to some integral $\bar{I}$. We omit the proof, which consists of an easy verification. An important consequence of eqs. (20) and (21) is that, if we have a transformation like that in eq. (20), *i.e.* a purely spatial transformation, but with $f = \bar{f} \neq 0$, we can obtain another transformation, corresponding to the same integral $\bar{I}$, involving also the time but with vanishing gauge function. This can be accomplished by choosing

(22) $$\tau = -\,\frac{\bar{f}}{\mathscr{L}}\,.$$

Thus, a purely spatial quasi-symmetry can always be changed into a space-time true symmetry. It is also easy to verify that the inverse theorem and the above proposition can be extended to nonconservative systems.

3. – The Noether theorem in the space of events

Let us reformulate the Noether theorem in the homogeneous Lagrangian variables [7]. Therefore, let us consider the space with $N+1$ dimensions (the *"space of events"* of Synge or the *"film-space"* of Schouten) given by the N Lagrangian coordinates q^i and the time t. Let x^α be the coordinates in this space, with $x^0 = t$ and $x^i = q^i$ so that, from here on, the Greek indices run from 0 to N. A curve Γ, in this space

$|\mathbf{R}\times\mathbf{Q}$, where $\mathbf{Q}$ is the ordinary configuration space, will be denoted by the equations

$$x^\alpha = x^\alpha(w)\,, \tag{23}$$

where the x^α are supposed C^2 functions and their derivatives

$$x'^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}w}\,, \tag{24}$$

not all simultaneously vanishing for the considered values of the independent parameter w, which can be completely arbitrary if we define a new Lagrangian

$$\Lambda \equiv \Lambda(x^0, x^1, \dots, x^N; x'^0, x'^1, \dots, x'^N)\,, \tag{25}$$

which is a positive homogeneous function of first degree in the "velocities" x'^α [8]. In this circumstance, the Hamilton variational principle will give equations of motion equivalent to those obtained from the standard Lagrangian (10). From (1) and (25), we have in fact

$$\Lambda(x^0, x^1, \dots, x^N; x'^0, x'^1, \dots, x'^N) = \mathscr{L}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t))\,\frac{\mathrm{d}t}{\mathrm{d}w} = \mathscr{L}\left(x^0, x^1, \dots, x^N; \frac{x'^1}{x'^0}, \frac{x'^2}{x'^0}, \dots, \frac{x'^N}{x'^0}\right)x'^0. \tag{26}$$

Due to the homogeneity condition, the Euler theorem gives

$$\frac{\partial\Lambda}{\partial x'^\alpha}x'^\alpha = \Lambda\,, \tag{27}$$

so that we may say that the new Hamiltonian is identically zero. Defining the action integral

$$I(\Gamma) = \int_{w_1}^{w_2}\Lambda(x^\alpha, x'^\alpha)\,\mathrm{d}w\,, \tag{28}$$

the variational equation $\delta I = 0$ will be equivalent to the Lagrangian system

$$\frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{\partial\Lambda}{\partial x'^\alpha}\right) - \frac{\partial\Lambda}{\partial x^\alpha} = 0\,. \tag{29}$$

The system (29) consists of $N+1$ equations, but only N of them are independent because, due to eq. (27), from (29) we have

$$x'^\alpha\left(\frac{\mathrm{d}}{\mathrm{d}w}\frac{\partial\Lambda}{\partial x'^\alpha} - \frac{\partial\Lambda}{\partial x^\alpha}\right) = \frac{\mathrm{d}}{\mathrm{d}w}\left(x'^\alpha\frac{\partial\Lambda}{\partial x'^\alpha}\right) - x''^\alpha\frac{\partial\Lambda}{\partial x'^\alpha} - x'^\alpha\frac{\partial\Lambda}{\partial x^\alpha} = \frac{\mathrm{d}\Lambda}{\mathrm{d}w} - \frac{\mathrm{d}\Lambda}{\mathrm{d}w} = 0\,, \tag{30}$$

so that system (29), with the constraint of eq. (30), is equivalent to that of eq. (10).

Starting from the Lagrangian (25) and the action integral (28), we can now reformulate the Noether theorem (or, rather, the conditions which must be satisfied in order that the transformations under consideration be Noetherian) in the space of events. We consider transformations

$$\bar{x}^\alpha = x^\alpha + \varepsilon\xi^\alpha(x, x'), \qquad \bar{x}'^\alpha = x'^\alpha + \varepsilon\xi'^\alpha(x, x'), \qquad \alpha, \beta = 0, 1, 2, \ldots, N. \tag{31}$$

Being the action integral independent of the choice of the parameter, due to homogeneity of the Lagrangian, there is obviously no need to consider transformations of the parameter. If we now require that the action integral (28) be invariant, up to a gauge term, under the transformations (31), we get

$$\frac{\partial\Lambda(x, x')}{\partial x^\alpha}\,\delta x^\alpha + \frac{\partial\Lambda(x, x')}{\partial x'^\alpha}\,\delta x'^\alpha = \varepsilon\,\frac{\mathrm{d}f(x, x')}{\mathrm{d}w}, \tag{32}$$

where

$$\begin{cases} \delta x^\alpha = \varepsilon\xi^\alpha(x, x'), \\ \delta x'^\alpha = \varepsilon\xi'^\alpha(x, x') = \varepsilon\left(\dfrac{\partial\xi^\alpha}{\partial x^\beta}x'^\beta + \dfrac{\partial\xi^\alpha}{\partial x'^\beta}x''^\beta\right) \end{cases} \tag{33}$$

and $f(x^\beta, x'^\beta)$ is the gauge function. By substituting, we have immediately

$$\xi^\alpha\,\frac{\partial\Lambda(x, x')}{\partial x^\alpha} + \frac{\partial\Lambda(x, x')}{\partial x'^\alpha}\left(\frac{\partial\xi^\alpha}{\partial x^\beta}x'^\beta + \frac{\partial\xi^\alpha}{\partial x'^\beta}x''^\beta\right) = \frac{\partial f}{\partial x^\alpha}x'^\alpha + \frac{\partial f}{\partial x'^\alpha}x''^\alpha. \tag{34}$$

Here ξ^α, Λ and f are all functions of x^α and x'^α, but not of x''^α, so that the equations

$$\begin{cases} \dfrac{\partial\Lambda}{\partial x'^\alpha}\,\dfrac{\partial\xi^\alpha}{\partial x'^\beta} = \dfrac{\partial f}{\partial x'^\beta}, \\ \dfrac{\partial\Lambda}{\partial x^\alpha}\,\xi^\alpha + \dfrac{\partial\Lambda}{\partial x'^\alpha}\,\dfrac{\partial\xi^\alpha}{\partial x^\beta}x'^\beta = \dfrac{\partial f}{\partial x^\alpha}x'^\alpha \end{cases} \tag{35}$$

must be separately true. In analogy to what done in the previous section, we call the system (35) *generalized Killing equations*. It is straightforward to verify that, substituting in eq. (34) the equations of motion (29), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{\partial\Lambda}{\partial x'^\alpha}\right)\xi^\alpha + \frac{\partial\Lambda}{\partial x'^\alpha}\left(\frac{\partial\xi^\alpha}{\partial x^\beta}x'^\beta + \frac{\partial\xi^\alpha}{\partial x'^\beta}x''^\beta\right) - \frac{\mathrm{d}f}{\mathrm{d}w} = 0, \tag{36}$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}w}\left(\frac{\partial\Lambda}{\partial x'^\alpha}\,\xi^\alpha - f\right) = 0. \tag{37}$$

Since it is easy to verify, from the definition (26), that

$$\text{(38)} \qquad \begin{cases} \dfrac{\partial \Lambda}{\partial x'^0} = \mathscr{L} - \dfrac{\partial \mathscr{L}}{\partial \dot{q}^i}\,\dot{q}^i = -\mathscr{H}\,, \\ \dfrac{\partial \Lambda}{\partial x'^i} = \dfrac{\partial \mathscr{L}}{\partial \dot{q}^i} = p_i\,, \end{cases}$$

eq. (37) coincides with eq. (11) because

$$\text{(39)} \qquad \frac{\partial \Lambda}{\partial x'^\alpha}\,\xi^\alpha - f = p_i\,\xi^i - \mathscr{H}\,\xi^0 - f = \frac{\partial \mathscr{L}}{\partial \dot{q}^i}\,\xi^i - \tau\left(\mathscr{L} - \frac{\partial \mathscr{L}}{\partial \dot{q}^i}\,\dot{q}^i\right) - f\,,$$

having posed $\xi^0 = \tau$. So, in place of eq. (16), we will have now

$$\text{(40)} \qquad I = \frac{\partial \Lambda(x, x')}{\partial x'^\alpha}\,\xi^\alpha(x, x') - f(x, x')\,.$$

It is also straightforward to reformulate in this context the inverse Noether theorem and its extension to systems with nonconservative forces. We omit them for the sake of brevity. We limit ourselves to notice that, if nonconservative forces Q_i are present as in eq. (13), the transformation of the generalized velocities will be

$$\text{(41)} \qquad \overline{x}'^0 = x'^0 + \varepsilon\xi'^0(x, x')\,, \qquad \overline{x}'^i = x'^i + \varepsilon[\xi'^i(x, x') + \Phi^i(x, x')\,x'^0]\,,$$

where the functions Φ^i must satisfy the identity

$$\text{(42)} \qquad \frac{\partial \Lambda}{\partial x'^i}\,\Phi^i = Q_i\,\xi^i\,.$$

4. – Killing equations in the Jacobi metric

For a Riemannian manifold of given metric, a vector **v** on it is a Killing vector if the Lie derivative of the metric along the vector **v** vanishes [9]. In other words, one can also say that **v** is a Killing vector for a given Riemannian manifold if on it the symmetrized covariant derivative $v_{(a|b)}$ results to be equal to zero [9] (Killing equations). This last definition can be immediately extended to a tensor of any order. A fundamental property of the Killing vectors is that their inner product with a tangent vector to the geodesics of the manifold is constant along the geodesics themselves. In general, the property also holds that contracting a Killing tensor of rank p by p tangent vectors one obtains a quantity which results to be constant on the geodesics [3], that is a first integral of the geodesic equations. Therefore, if we can find a Riemannian manifold whose geodesic lines coincide with the actual paths of a given dynamical system, we can establish a direct correspondence between the first integrals involving the Killing vectors or tensors on the geodesics and the first integrals of the dynamical system. The problem is then the "geometrization" of the dynamical system and, subsequently, to solve the relevant Killing equations.

Several techniques have been so far devised to attack the Killing equations, essentially in the context of General Relativity [10]. Applying conformal transformations, a complete study of the two-dimensional case has been performed recently [11], examining both the cases at arbitrary energy and at fixed energy. In the present section we want to stress the light that can be shed on classical systems by the use of these tools, with possible new consequences in the understanding of the integrable systems. We examine firstly the case of natural conservative systems, for which the geometrization based on the so-called Jacobi metric is particularly useful.

We are given a *natural* system, that is one for which the kinetic energy is given by

(43) $$\mathcal{T} = \frac{1}{2}\gamma_{ab}(\mathbf{q})\,\dot{q}^a\dot{q}^b, \qquad a, b = 1, \dots, N .$$

Be h the total (conserved) energy of the system, so that

(44) $$h = \mathcal{T} + V(\mathbf{q}),$$

being $V(\mathbf{q})$ the potential energy. Equation (43), as we know, defines *naturally* a Riemannian metric [5]

(45) $$\mathrm{d}s^2 = \gamma_{ab}\,\mathrm{d}q^a\,\mathrm{d}q^b$$

on the configuration space. Starting from this metric, the *Jacobi metric* [12] can then be defined:

(46) $$\mathrm{d}s_J^2 = g_{ab}\,\mathrm{d}q^a\,\mathrm{d}q^b,$$

with

(47) $$g_{ab} = 2(h - V(\mathbf{q}))\,\gamma_{ab}.$$

As is known, the geodesics of the Riemannian manifold endowed with the metric g_{ab} coincide with the physical path of the system with Lagrangian

(48) $$\mathcal{L} = \mathcal{T} - V(\mathbf{q})$$

and total energy h.

Let now ξ^a represent a *Killing vector* for the Jacobi metric, that is a vector such as to satisfy

(49) $$\xi_{(a|b)} = g_{(ac}\,\xi^c{}_{,b)} + g_{(ac}\,\Gamma^c{}_{b)d}\,\xi^d = 0\ ,$$

where the small vertical bar denotes the covariant derivative in the metric with coefficients g_{ab}, the Γ^a_{bc} are the Christoffel symbols related to g_{ab} and the round brackets denote the symmetrization of the indices. Equation (49) can be written in the form of eq. (9):

(50) $$g_{ab,c}\,\xi^c + g_{ac}\,\xi^c{}_{,b} + g_{cb}\,\xi^c{}_{,a} = 0\ ,$$

where the comma stays for the ordinary derivative. Putting eq. (47) into (50) we obtain

(51) $$(h - V)(\gamma_{ac}\,\xi^c_{,b} + \gamma_{bc}\,\xi^c_{,a} + \gamma_{ab,c}\,\xi^c) - \gamma_{ab}\,V_{,c}\,\xi^c = 0\ .$$

Equations (51) represent the conditions (expressed by means of the coefficients of the standard metric $\boldsymbol{\gamma}$ and the potential) which the components of the vector $\boldsymbol{\xi}$ must satisfy in order that $\boldsymbol{\xi}$ be a Killing vector for the Riemannian manifold endowed with the Jacobi metric g (*Jacobi configuration manifold* or, simply, Jacobi manifold). If $\boldsymbol{\xi}$ is a Killing vector for the Jacobi manifold, the quantity

$$I_1 = \xi_a \frac{\mathrm{d}q^a}{\mathrm{d}s_{\mathrm{J}}} ,$$

is constant along its geodesics [3] and then also on the physical paths of the system with Lagrangian $\mathscr{L}$ and total energy h. Therefore I_1 is a first integral of our dynamical system. Recalling now the discussion of the Noether theorem of sect. **2**, we see that the Killing vector $\boldsymbol{\xi}$ can be identified with the generator of the transformation (8) and the quantity I_1 can be identified with the quantity I in eq. (12), so that, at least from the formal point of view, the two approaches are unified in the vector case. It must be stressed that, from here on, we shall be concerned only with transformations characterized by $\tau(t, \mathbf{q}, \dot{\mathbf{q}}) \equiv 0$, since in the Jacobi form of the least- action principle the time has been eliminated. Since the geodesics on the Jacobi manifold correspond to the orbits of the representative point of the system at energy h in the configuration space (with $\boldsymbol{\gamma}$ metric), then I_1 is a first integral of the system at fixed energy. From the definition of $\mathrm{d}s_J$, the first integral has also the form

$$I_1 = \xi_a \frac{\mathrm{d}q^a}{\mathrm{d}s_{\mathrm{J}}} = g_{ab}\,\xi^b \frac{\mathrm{d}q^a}{\mathrm{d}s_{\mathrm{J}}} = \gamma_{ab}\,\xi^b \frac{\mathrm{d}q^a}{\mathrm{d}t} = \gamma_{ab}\,\xi^b \dot{q}^a = \xi^a p_a , \tag{52}$$

where the momentum p_a, since we are dealing with a natural system, is related to the velocity by the duality relation $p_a = \gamma_{ab}\dot{q}^b$. Moreover, if we assume the momenta defined through their covariant components

$$p_a = \frac{\partial \mathscr{L}}{\partial \dot{q}^a} = \gamma_{ab}\dot{q}^b , \tag{53}$$

we have that the covariant components are the same in both metrics ($\boldsymbol{\gamma}$ and $\mathbf{g}$) whereas, obviously, the contravariant components (that is the velocities) are different. The same will happen for $\boldsymbol{\xi}$, which we assume to be introduced through its contravariant components.

If we want that I_1 is a first integral of the system at arbitrary energy (as is usually required in classical mechanics), eqs. (51) *split* into the two independent equations

$$\gamma_{ac}\,\xi^c_{,\,b} + \gamma_{bc}\,\xi^c_{,\,a} + \gamma_{ab,\,c}\,\xi^c = 0 , \qquad V_{,\,c}\,\xi^c = 0 , \tag{54}$$

that we can also write as

$$\xi_{(a;\,b)} = 0 , \qquad V_{,\,c}\,\xi^c = 0 , \tag{55}$$

where the semicolon denotes the covariant derivative performed with respect to the $\boldsymbol{\gamma}$ metric. Formulas (55) can be *read* in the following alternative way: the vectors ξ^a that, beside being Killing vectors for the $\boldsymbol{\gamma}$ metric, satisfy also the condition $V_{,\,c}\,\xi^c = 0$, give rise to first integrals linear in the momenta. A simple but fundamental solution of eqs. (55) is that referring to spherical symmetric potentials. In two dimensions (the treatment is valid for an arbitrary dimension) we can write

$$\xi^1 = A\,, \qquad \xi^2 = B\,, \qquad I_1 = Ap_x + Bp_y, \tag{56}$$

so that eqs. (55) reduce to

$$A_{,x} = B_{,y} = 0\,, \qquad A_{,y} + B_{,x} = 0\,, \qquad AV_{,x} + BV_{,y} = 0\,. \tag{57}$$

A solution of this last system is

$$A = -y\,, \qquad B = x\,, \qquad V = V(r)\,, \qquad r = \sqrt{x^2 + y^2}, \tag{58}$$

so that the Killing vector is

$$\boldsymbol{\xi}_c = \begin{pmatrix} -y \\ x \end{pmatrix} \tag{59}$$

and the invariant is

$$I_1 = xp_y - yp_x, \tag{60}$$

that is the angular momentum, which is conserved in the potentials of the form $V = V(r)$. From the point of view of the Noether theorem, it corresponds to a transformation of the kind (8) given by

$$\bar{x} = x - \varepsilon y\,, \qquad \bar{y} = y + \varepsilon x\,, \tag{61}$$

that is to an infinitesimal rotation in the (x, y)-plane.

In complete analogy with what obtained in the Killing-vectors case, we have that associated to *M-rank Killing tensors*, that is completely symmetric tensors satisfying the equations

$$K_{(abc\ldots|d)} = 0\,, \tag{62}$$

there are integrals of motion of degree M in the momenta (see, *e.g.*, Eisenhart [3]),

$$J_M = K^{abc\ldots} p_a p_b p_c \ldots\,. \tag{63}$$

There is, however, an important difference with respect to what seen about Killing vectors: the form of the invariant in the Jacobi metric J_M and that of the same invariant in the corresponding physical system is different, at least formally, since the relation between the Killing tensor in the Jacobi metric (hereinafter referred to as the *Jacobi gauge*) and that in the γ metric (hereinafter referred to as the *physical gauge*) is not trivial. For systems with Hamiltonian of well-defined time-reversal symmetry it can be proven [13] that the invariant must be either even or odd in the momenta, a fact that allows a considerable simplification of the treatment. For example, the most general form of the second-order invariant in the physical gauge is

$$I_2 = A^{ab} p_a p_b + \Phi\,, \tag{64}$$

where Φ is a function on the configuration space and the tensor **A**, together with the Killing tensor **K**, is introduced through its contravariant components.

On the other hand, from the Noether theorem, using (11) with $\tau \equiv 0$, we have

$$I = \xi^a p_a - f \,, \tag{65}$$

where the gauge function f depends not only on the coordinates but also on the velocities (or the momenta). The same will happen for the generator ξ^a, that cannot be anymore considered as a vector field on the configuration manifold but must be treated as a generalized symmetry generator acting on the tangent bundle (see, *e.g.*, [14]). Therefore, it will be possible to compare the two forms (64) and (65) only in specific examples rather than in general terms.

Given the form (64) of the invariant, it is straigthforward to find the conditions that must be satisfied to grant its existence. The Poisson bracket of this invariant with the Hamiltonian is

$$[I_2, H] = \gamma^{ab} p_b (A^{cd}{}_{,a} p_c p_d + \Phi_{,a}) - 2A^{ab} p_b V_{,a} . \tag{66}$$

The vanishing of the Poisson bracket for every value of the momenta implies the set of equations

$$A_{(ab;c)} = 0 \,, \qquad 2A_a^b V_{,b} = \Phi_{,a} . \tag{67}$$

The likelihood of this system of equations with eqs. (55) obtained in the Killing-vector case is soon recognized: we see that A^{ab} must be a Killing tensor of the $\boldsymbol{\gamma}$ metric and that the first derivatives of the potential and of the Φ function must be related to it in a suitable way. The problem now is that we cannot simply write

$$A_{(ab|c)} = 0 \tag{68}$$

as the general Jacobi gauge condition for the existence of the invariant for the very reason that it does not account for an inhomogeneous invariant of the form (64). However, the incompatibility between the two sets of equations is easily "cured" by exploiting a simple property of the Jacobi geometry. In the chain of equalities (15) it is implicitly assumed the existence of a *Jacobi velocity vector*

$$u^a = \frac{\mathrm{d}q^a}{\mathrm{d}s_\mathrm{J}} = \frac{1}{2(h - V)} \frac{\mathrm{d}q^a}{\mathrm{d}t} . \tag{69}$$

Moreover it will be

$$p_a = \gamma_{ab} \dot{q}^b = 2(E - V) \gamma_{ab} \frac{\mathrm{d}q^b}{\mathrm{d}s_\mathrm{J}} = g_{ab} \frac{\mathrm{d}q^b}{\mathrm{d}s_\mathrm{J}} . \tag{70}$$

That is, the momenta (or rather, their covariant components) in the Jacobi gauge are in the duality relationship with the "Jacobi velocities" in the same way as they are with the physical velocities in the physical gauge (in each gauge by means of the relevant metric tensor). Moreover, from the very definition of the Jacobi metric (47), we have that both the Jacobi velocity and the conjugate momentum are unit vectors,

$$g_{ab} \frac{\mathrm{d}q^a}{\mathrm{d}s_\mathrm{J}} \frac{\mathrm{d}q^b}{\mathrm{d}s_\mathrm{J}} = g^{ab} p_a p_b = 1 . \tag{71}$$

To connect the two expressions of the invariant in the two gauges we perform then the following simple "trick": in (64) we multiply the inhomogeneous term Φ by $g^{ab} p_a p_b$ (which is 1, as we have just seen) and, by factoring out the momenta, we can rewrite eq. (64),

$$I_2 = (A^{ab} + g^{ab}\Phi)\, p_a p_b , \tag{72}$$

so that, comparing this with eq. (63) specialized to the case of a second-order invariant, we can make the position

$$K^{ab} = A^{ab} + g^{ab}\Phi = A^{ab} + \frac{\gamma^{ab}}{2(h-V)}\Phi . \tag{73}$$

Let us see that this position is consistent, showing that if the tensor defined in (73) satisfies the Killing-tensor equation, we obtain also the equations for the invariant in the physical gauge (67). Equations (62) in the case of a rank-2 tensor are

$$K^{(cd}{}_{;\, b)} = K^{(cd}{}_{,\, b)} + \Gamma^{(c}{}_{ba} K^{ad)} + \Gamma^{(d}{}_{ba} K^{c)a} = 0 , \tag{74}$$

which, in the Jacobi metric case, become

$$(h-V)\, \gamma_{(cm}\gamma_{dl} K^{ml}{}_{,\, b)} - \gamma_{(bc}\gamma_{d)l} K^{al} V_{,\, a} = 0 . \tag{75}$$

Putting in these equations the definition (73) of the Killing tensor and noting that

$$K^{ml}{}_{,\, b} = A^{ml}{}_{,\, b} + \frac{V_{,\, b}\Phi}{2(h-V)^2}\gamma^{ml} + \frac{\Phi_{,\, b}}{2(h-V)}\gamma^{ml} , \tag{76}$$

we obtain the equations

$$(h-V)\, \gamma_{(cm}\gamma_{dl} A^{ml}{}_{,\, b)} + \frac{1}{2}\gamma_{(cd}\Phi_{,\, b)} - \gamma_{(bc}\gamma_{d)l} K^{al} V_{,\, a} = 0 . \tag{77}$$

If these last equations must be valid for arbitrary values of the energy, they split into two sets of equations that coincide with (67). To be fully consistent in all the treatment, it must be remarked that the form given for the Killing-tensors equations in both gauges (eqs. (62) and (67)) contains the covariant form of the second-order Killing tensor, so that, in order to check explicitly their expressions, also the fully covariant analogous of the position (73) is needed:

$$K_{ab} = 4(h-V)^2 A_{ab} + g_{ab}\Phi . \tag{78}$$

A dedicated study requires the more general situation in which one is interested in the existence of invariants at given fixed values of the energy (the so-called *configurational invariants*), both in the Killing-vector and in the Killing-tensor case. The unified geometric approach to configurational and arbitrary energy invariants in systems with two degrees of freedom is given in ref. [11].

5. – Killing tensors in two-dimensional systems

Let us consider the case of a system with two degrees of freedom so that the invariant in the physical gauge, using Cartesian coordinates, can be written in the form

$$I_2 = Ap_x^2 + 2Bp_x p_y + Cp_y^2 + \Phi \,, \tag{79}$$

where we have written the components of the Killing tensor in the physical gauge as

$$A_{11} = A \,, \qquad A_{12} = B \,, \qquad A_{22} = C \,. \tag{80}$$

In this case

$$\gamma_{ab} = \eta_{ab} \,, \tag{81}$$

the Killing-tensor equations $A_{(ab;\,c)} = 0$ turn out to be

$$\begin{cases} A_{,\,x} = 0 \,, & 2B_{,\,y} + C_{,\,x} = 0 \,, \\ C_{,\,y} = 0 \,, & 2B_{,\,x} + A_{,\,y} = 0 \,, \end{cases} \tag{82}$$

whose solutions are

$$\begin{cases} A = \dfrac{1}{2} ay^2 + cy + e \,, \\ B = -\dfrac{1}{2}(axy + cx + by - g) \,, \\ C = \dfrac{1}{2} ax^2 + bx + d \,, \end{cases} \tag{83}$$

where a, b, c, d, e, g are arbitrary constants. Collecting this solution in a compact matrix form we can write it as

$$\begin{aligned} \mathbf{A}^{(2)} &= \frac{1}{2} \begin{pmatrix} ay^2 + 2cy + 2e & -axy - cx - by + g \\ -axy - cx - by + g & ax^2 + 2bx + 2d \end{pmatrix} \\ &= a\mathbf{M} + b\mathbf{L}_1 + c\mathbf{L}_2 + e\mathbf{E}_1 + d\mathbf{E}_2 + g\mathbf{E}_3 \,, \end{aligned} \tag{84}$$

where we introduced the tensors

$$\mathbf{M} = \frac{1}{2} \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}, \qquad \mathbf{L}_1 = \frac{1}{2} \begin{pmatrix} 0 & -y \\ -y & 2x \end{pmatrix}, \qquad \mathbf{L}_2 = \frac{1}{2} \begin{pmatrix} 2y & -x \\ -x & 0 \end{pmatrix} \tag{85}$$

and

$$\mathbf{E}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{E}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{E}_3 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{86}$$

that are the Killing tensors of the Euclidean metric in two dimensions. Actually, by a simple inspection, it is easy to see that all of them can be expressed as tensor product of the Killing vectors of the plane. This is an example of a more general result, due to Katzin and Levine ([15]), that in every space with constant curvature the Killing tensors are *reducible*. It is clear, therefore, that in the case of a homogeneous invariant (that is when $\Phi = 0$ in eq. (79)), we have again the law of conservation of the angular momentum. However, when the invariant is non-homogeneous and its quadratic part is associated to the tensor **M**, we have the non-trivial case of the well-known *Stäckel potentials* (to see this we refer to [11]).

6. – Classical cases

It is particularly useful to apply the results obtained so far to the already known classical integrable cases, since, even in these contexts it is possible to look at the structure of the dynamical systems from a completely geometrical point of view: this can provide indication on the path to follow in other cases not yet investigated. But, before passing to consider examples of Killing tensors associated to the Jacobi geometry of some classical two-dimensional case, we want to give an example of the proposition we have enunciated at the end of sect. **2**.

6˙1. *The energy integral.* – It is easy to check that the transformation defined by

$$\tau = \text{const} \equiv -1\,, \qquad \xi^i \equiv 0 \quad (\forall i)\,, \qquad f \equiv 0 \tag{87}$$

gives, for (16), $I = \mathscr{H}$, that is the conservation of energy. On the other hand, if for the same system we consider the transformation given by

$$\tau \equiv 0\,, \qquad \bar{\xi}^i \equiv \dot{q}^i\,, \tag{88}$$

we obtain, by $\bar{f} = \mathscr{L}$, again the conservation of energy. We have, then, an example that the transformation corresponding to a given first integral is not unique. By applying eq. (21) with $\tau = -1$, we obtain in fact

$$\begin{cases} f = \mathscr{L} - \mathscr{L} = 0\,, \\ \xi^1 = \dot{q}^1 - \dot{q}^1 = 0\,, \\ \xi^2 = \dot{q}^2 - \dot{q}^2 = 0\,, \end{cases} \tag{89}$$

which is just the transformation considered above.

6˙2. *The Laplace-Runge-Lenz vector.* – Among the classical integrable potentials, the *Kepler potential*

$$V = -\frac{\mu}{r}\,, \qquad \mu > 0\,, \qquad r = \sqrt{x^2 + y^2}\,, \tag{90}$$

plays a central role: it is the prototype case in which a degeneracy implies the separability of the dynamical equations in more than one coordinate system. This fact in turn implies the existence of additional independent invariants. From the perspective of the present paper this means the existence of several Killing tensors associated to the *Jacobi-Kepler* geometry,

$$g_{ab} = 2\left(h + \frac{\mu}{r}\right)\eta_{ab}\,. \tag{91}$$

The Kepler potential is obviously in the class of the solution (58) of the equations for the linear invariant, which is then the angular momentum of eq. (59). Regarding the Killing tensor **M**, the discussion at the end of the previous section holds. Let us see, now, what happens if we choose the two tensors $\mathbf{L}_1$ and $\mathbf{L}_2$. If these two tensors give indeed two invariants of the form of eq. (79)), then, recalling eqs. (67), the solutions of the equations

$$2V_{,b}L_1{}^b{}_a = \Phi_{1,a}, \qquad 2V_{,b}L_2{}^b{}_a = \Phi_{2,a}, \tag{92}$$

must exist, where $V = -\mu/r$. It is easy to check that two simple solutions of this system are

$$\Phi_1 = -\mu\frac{x}{r}\,, \qquad \Phi_2 = -\mu\frac{y}{r} \tag{93}$$

so that, collecting the results, we can write the two invariants in the form

$$\begin{cases} I^{(1)}_{\mathrm{LRL}} = \dfrac{1}{\mu}(xp_y^2 - yp_xp_y) - \dfrac{x}{r}\,, \\ I^{(2)}_{\mathrm{LRL}} = \dfrac{1}{\mu}(yp_x^2 - xp_xp_y) - \dfrac{y}{r}\,, \end{cases} \tag{94}$$

where, actually, we have divided both expressions by μ. These two quadratic invariants are the two components of the *Laplace-Runge-Lenz vector* that gives the fixed orientation of the Kepler ellipse. From the point of view of the Noether theorem, we know that the two first integrals of eqs. (94) correspond to generators of the form

$$\xi_a = A_{abc}\dot{q}^b q^c, \qquad \tau \equiv 0 \qquad (A_{abc} = \mathrm{const}, \quad \forall a, b, c)\,. \tag{95}$$

If we call $\boldsymbol{\xi}^{(1)}$ and $\boldsymbol{\xi}^{(2)}$ the generators in the two cases of eqs. (94), their components and the gauge functions are, respectively,

$$\begin{cases} \xi_x{}^{(1)} = -\dfrac{1}{\mu}yp_y, \\ \xi_y{}^{(1)} = \dfrac{1}{\mu}(2xp_y - yp_x)\,, \\ f^{(1)} = \dfrac{1}{\mu}(xp_y^2 - yp_xp_y) + \dfrac{x}{r} \end{cases} \tag{96}$$

and

$$\left\{\begin{aligned} \xi_x^{(2)} &= \frac{1}{\mu}(2yp_x - xp_y),\\ \xi_y^{(2)} &= -\frac{1}{\mu}xp_x,\\ f^{(2)} &= \frac{1}{\mu}(yp_x^2 - xp_xp_y) + \frac{y}{r},\end{aligned}\right. \tag{97}$$

with a suitable choice of the arbitrary constants. From these results the different role of the inhomogeneous part Φ and the gauge functions is evident.

Using the relation between the Killing tensors in the two gauges (73), we can write now the expressions of the Jacobi-Killing tensors for the two components of the Laplace-Runge-Lenz vector. In this case we have

$$K_{\mathrm{LRL}}^{(1,\,2)ab} = \frac{1}{\mu} L_{1,\,2}{}^{ab} + \frac{\eta^{ab}}{\mu(h+\mu/r)}\Phi_{1,\,2}, \tag{98}$$

or, explicitly,

$$\mathbf{K}_{\mathrm{LRL}}^{(1)} = \frac{1}{\mu}\begin{pmatrix} -\dfrac{\mu x}{2(hr+\mu)} & -\dfrac{y}{2}\\ -\dfrac{y}{2} & x\left(1-\dfrac{\mu}{2(hr+\mu)}\right)\end{pmatrix} \tag{99}$$

and

$$\mathbf{K}_{\mathrm{LRL}}^{(2)} = \frac{1}{\mu}\begin{pmatrix} y\left(1-\dfrac{\mu}{2(hr+\mu)}\right) & -\dfrac{x}{2}\\ -\dfrac{x}{2} & -\dfrac{\mu y}{2(hr+\mu)}\end{pmatrix}. \tag{100}$$

If we denote with c the value of the angular-momentum invariant of eq. (59), it is easy to check the relation

$$(I_{\mathrm{LRL}}^{(1)})^2 + (I_{\mathrm{LRL}}^{(2)})^2 = 1 + \frac{2}{\mu^2}hc^2. \tag{101}$$

6˙3. *The harmonic oscillator.* – Let us consider now the other fundamental integrable classical potential: that of the *harmonic oscillator*. Since we have developed all the above treatment for two-dimensional systems, we will continue studying two-dimensional oscillators, but we have to remark that, at least in the case of the *isotropic* oscillator, the results that will be illustrated in the following remain true in an arbitrary number of dimensions.

We have then the potential

$$V = \frac{1}{2}(\omega_x^2 x^2 + \omega_y^2 y^2)\,, \tag{102}$$

so that the Jacobi geometry in this case is

$$g_{ab} = (2h - \omega_x^2 x^2 - \omega_y^2 y^2)\,\eta_{ab}\,. \tag{103}$$

Let us examine, for the moment, the isotropic oscillator:

$$\omega_x = \omega_y = \omega\,. \tag{104}$$

Besides the angular-momentum invariant of eq. (59), the existence of the three *Runge-type* quadratic invariants is well known:

$$\begin{cases} I_{\rm IO}^{(1)} = \dfrac{1}{2}(p_x^2 + \omega^2 x^2)\,,\\[2mm] I_{\rm IO}^{(2)} = \dfrac{1}{2}(p_x p_y + \omega^2 xy)\,,\\[2mm] I_{\rm IO}^{(3)} = \dfrac{1}{2}(p_y^2 + \omega^2 y^2)\,. \end{cases} \tag{105}$$

Recalling the three constant tensors of eq. (86), these three invariants can be written in the form

$$I_{\rm IO}^{(j)} = \frac{1}{2} E_j^{ab} p_a p_b + \Phi_{\rm IO}^{(j)}\,, \qquad j = 1, 2, 3\,, \tag{106}$$

where we have introduced the three functions

$$\begin{cases} \Phi_{\rm IO}^{(1)} = \dfrac{1}{2}\omega^2 x^2,\\[2mm] \Phi_{\rm IO}^{(2)} = \dfrac{1}{2}\omega^2 xy\,,\\[2mm] \Phi_{\rm IO}^{(3)} = \dfrac{1}{2}\omega^2 y^2. \end{cases} \tag{107}$$

On the other hand, we know that for the n-dimensional isotropic oscillator, to Noetherian transformations generated by

$$\xi_b = a_{bc}\dot{q}^c\,, \qquad \text{with } a_{bc} = a_{cb} = \text{const} \tag{108}$$

and

$$f(q, \dot{q}) = \frac{1}{2}(a_{bc}\dot{q}^b\dot{q}^c - \omega^2 a_{bc} q^b q^c)\,, \tag{109}$$

the $n(n+1)/2$ independent components of the generalized tensor

$$I^{bc} = \frac{1}{2}(\dot{q}^b \dot{q}^c + \omega^2 a_{bc} q^b q^c)\,, \tag{110}$$

correspond as constants of the motion. For $n = 2$ its three independent components (in Cartesian coordinates) are just those given by eqs. (105). The three corresponding gauge functions are

$$\begin{cases} f_1 = \dfrac{1}{2}(p_x^2 - \omega^2 x^2)\,, \\[2ex] f_2 = \dfrac{1}{2}(p_x p_y - \omega^2 xy)\,, \\[2ex] f_3 = \dfrac{1}{2}(p_y^2 - \omega^2 y^2)\,. \end{cases} \tag{111}$$

At last, the Jacobi-Killing tensors corresponding to eqs. (105) are, respectively,

$$\begin{cases} \mathbf{K}_{\rm IO}^{(1)} = \dfrac{1}{2}\begin{pmatrix} \dfrac{2h - \omega^2 y^2}{2h - \omega^2 r^2} & 0 \\[2ex] 0 & \dfrac{\omega^2 x^2}{2h - \omega^2 r^2} \end{pmatrix}, \\[6ex] \mathbf{K}_{\rm IO}^{(2)} = \dfrac{1}{2}\begin{pmatrix} \dfrac{\omega^2 xy}{2h - \omega^2 r^2} & \dfrac{1}{2} \\[2ex] \dfrac{1}{2} & \dfrac{\omega^2 xy}{2h - \omega^2 r^2} \end{pmatrix}, \\[6ex] \mathbf{K}_{\rm IO}^{(3)} = \dfrac{1}{2}\begin{pmatrix} \dfrac{\omega^2 y^2}{2h - \omega^2 r^2} & 0 \\[2ex] 0 & \dfrac{2h - \omega^2 x^2}{2h - \omega^2 r^2} \end{pmatrix}. \end{cases} \tag{112}$$

For the *anisotropic* oscillator ($\omega_x \neq \omega_y$) it is easy to check that the two invariant *energies*

$$\begin{cases} I_{\rm AO}^{(1)} = \dfrac{1}{2}(p_x^2 + \omega_x^2 x^2)\,, \\[2ex] I_{\rm AO}^{(3)} = \dfrac{1}{2}(p_y^2 + \omega_y^2 y^2) \end{cases} \tag{113}$$

are still conserved, but the invariant that generalizes $I_{\rm IO}^{(2)}$ does not always exist. It is nice then to look at the condition imposed on the frequencies by the compatibility equation of (67)

$$2A_a^{\ b}V_{,b}=\Phi_{,a}. \tag{114}$$

Choosing, for example, the tensor $\mathbf{L}_1$ we obtain the system (92) for the potential (102), namely

$$\begin{cases} -\omega_y^2 y^2=\Phi_{1,x}\,,\\ 2xy\left(\omega_y^{\ 2}-\dfrac{\omega_x^{\ 2}}{2}\right)=\Phi_{2,y}; \end{cases} \tag{115}$$

the compatibility conditions have the following solution:

$$\omega_x=2\omega_y=2\omega\,,\qquad \Phi_1=-\omega^2xy^2, \tag{116}$$

so that the corresponding invariant is

$$I_{\mathrm{AO}}^{(2)}=xp_y^{\ 2}-yp_xp_y-\omega^2xy^2. \tag{117}$$

Choosing, instead, the tensor $\mathbf{L}_2$ the compatibility conditions have now the solution

$$\omega_y=2\omega_x=2\omega\,,\qquad \Phi_2=-\,\omega^2x^2y\,. \tag{118}$$

For this last case, the corresponding Noetherian transformation has the generators of the type

$$\xi_a=A_{abc}\dot{q}^bq^c,\qquad \tau\equiv 0\quad (A_{abc}=\mathrm{const}\quad \forall a,\,b,\,c)\,. \tag{119}$$

which (since $\omega_y=2\omega_x=2\omega$) result to be

$$\begin{cases} \xi_1=-\,2yp_x+xp_y=-yp_x+c\,,\\ \xi_2=xp_x, \end{cases} \tag{120}$$

with

$$f=xp_xp_y-yp_x^2-\omega^2x^2y\,. \tag{121}$$

Generalizing these results, it can be said that in the case of commensurability ratios greater than 2 between the frequencies, we would find that the Killing and compatibility equations could be solved by higher-order Killing tensors.

6$\cdot$4. *The generalized Hénon-Heiles case.* – Let us consider the *generalized Hénon-Heiles potential*

$$V=\frac{1}{2}(\omega_x^{\ 2}x^2+\omega_y^{\ 2}y^2)+x^2y+\frac{\mu}{3}y^3, \tag{122}$$

so called because it generalizes the famous potential introduced by Hénon and Heiles [16], that is a particular case of the above potential when the parameter μ is equal to -1. We try to solve the system (92), inserting in it the various Killing tensors we know of (eqs. (85) and (86)).

Let us start with the constant tensors of eq. (86). It is easy to see that with the tensors $\mathbf{E}_1$ and $\mathbf{E}_2$ the compatibility equations for the function Φ cannot be solved. We see instead that we can find a meaningful solution using $\mathbf{E}_3$. In fact from the system

(123) $$2E_{3a}{}^{b}V_{,b}=\Phi_{,a}$$

we obtain the two equations

(124) $$\begin{cases}\Phi_{,x}=\omega_y^2 y+x^2+\mu y^2,\\ \Phi_{,y}=\omega_x^2 x+2xy\,.\end{cases}$$

From this, the condition

(125) $$\Phi_{,xy}=\Phi_{,yx}$$

imposes the constraints

(126) $$\omega_x=\omega_y=\omega\,,\qquad \mu=1\,,$$

so that, integrating, we find

(127) $$\Phi\equiv\Phi_*=\omega^2xy+xy^2+\frac{1}{3}x^3$$

and, collecting the results, the second invariant in the case of a potential of the form

(128) $$V=\frac{1}{2}\omega^2(x^2+y^2)+x^2y+\frac{1}{3}y^3,$$

is

(129) $$I=E_3^{ab}p_ap_b+\Phi_*=p_xp_y+\omega^2xy+xy^2+\frac{1}{3}x^3.$$

This result is obtained with a different procedure in the review by Hietarinta [13]. Furthermore, the solution of the generalized Killing equations (5) and (6) corresponding to this case is given by

(130) $$\xi_b=a_{bc}\dot{q}^c,\qquad \text{with } a_{12}=a_{21}=1\,,\qquad a_{11}=a_{22}=0$$

and

(131) $$f=p_xp_y-\omega^2xy-xy^2-\frac{1}{3}x^3.$$

One may wonder if some case in which

(132) $$\omega_x\neq\omega_y$$

does exist.

To this aim we may try with one of the tensors $\mathbf{L}_1$ or $\mathbf{L}_2$. Choosing $\mathbf{L}_2$, in place of the system (124), we get

$$\text{(133)} \qquad \begin{cases} \Phi_{,x} = y(\omega_x^2 x + 2xy) - \dfrac{x}{2}(\omega_y^2 y + x^2 + \mu y^2)\,, \\ \Phi_{,y} = -x(\omega_x^2 x + 2xy)\,. \end{cases}$$

In this case the condition (125) gives

$$\text{(134)} \qquad 2\omega_x = \omega_y\,, \qquad \mu = 6\,,$$

so that the second invariant is

$$\text{(135)} \qquad I = L_2^{ab} p_a p_b + \Phi = y p_x^2 - x p_x p_y - \omega_x^2 x^2 y - x^2 y^2 - \frac{1}{4} x^4 .$$

Is there the possibility of finding a second invariant without any restriction on the frequencies? It is not difficult to see that there is a very simple way to generalize the last result to arbitrary values of the frequencies if, as the tensor to put into the compatibility equations (92), we choose the combination

$$\text{(136)} \qquad \mathbf{A} = c_1 \mathbf{E}_1 + c_2 \mathbf{L}_2$$

with c_1 and c_2 constants to be determined in order to satisfy eqs. (125) and (132). In place of the system (133), we will have

$$\text{(137)} \qquad \begin{cases} \Phi_{,x} = c_1(\omega_x^2 x + 2xy) + c_2 y(\omega_x^2 x + 2xy) - \dfrac{c_2 x}{2}(\omega_y^2 y + x^2 + \mu y^2)\,, \\ \Phi_{,y} = -c_2 x(\omega_x^2 x + 2xy)\,. \end{cases}$$

It is easy to see that the condition (125) is satisfied if again $\mu = 6$, but the other constraint now is

$$\text{(138)} \qquad 4c_1 + c_2(4\omega_x^2 - \omega_y^2) = 0\,.$$

If, therefore we make the choice

$$\text{(139)} \qquad c_1 = 4\omega_x^2 - \omega_y^2,\; c_2 = -4\,,$$

we have a solution which is valid for any value of the frequencies. Collecting the results we see that the expression of the second invariant in this most general case is

$$\text{(140)} \qquad I = ((4\omega_x^2 - \omega_y^2)\, E_1^{ab} - 4L_2^{ab})\, p_a p_b + \Phi =$$

$$= (4\omega_x^2 - \omega_y^2 - 4)\, p_x^2 - 4x p_x p_y + 4\omega_x^2 x^2 y + (4\omega_x^2 - \omega_y^2)\,\omega_x^2 x^2 - 4x^2 y^2 - \frac{1}{4} x^4 .$$

This result was obtained in a different context by Greene as is quoted in [17].

6.5. *Remark.* – For the cases in the subsections from **6**.2 to **6**.4, owing to what we have said at the end of sect. **2**, it would be possible to obtain Noetherian transformations with $f=0$, but with $\tau \neq 0$. These transformations are considerably more complicated than the ones we have given and, in any case, their generators cannot be considered as Killing vectors, since they are depending also on the velocities.

7. – Killing equations in the Finsler metric

In this section we show how the connection between the variational symmetry of a Lagrangian problem, exploited by the Noether theorem, and the symmetry of a Riemannian metric (the Jacobi metric) can be generalized to non-conservative problems, at the price of the introduction of a non-Riemannian metric, the *Finsler metric*.

7.1. *The space of events as a Finsler space.* – We consider now a completely general system with monogenic forces, in which the kinetic energy can contain also terms linear in the velocities and the potential can be also explicitly time-depending:

$$\mathscr{L}(t, \mathbf{q}(t), \dot{q}(t)) = \mathscr{T}(\mathbf{q}(t), \dot{q}(t)) - V(t, \mathbf{q}). \tag{141}$$

According to the prescription introduced in sect. **2**, the homogeneous Lagrangian of eq. (25) is therefore

$$\Lambda(x, x') = [\mathscr{T}(\mathbf{q}(t), \dot{q}(t)) - V(t, \mathbf{q})] \frac{\mathrm{d}t}{\mathrm{d}w} . \tag{142}$$

In the particular case of a natural conservative system, Λ would be given by

$$\Lambda(x, x') = \frac{1}{2} \gamma_{ab}(x)\, x'^a x'^b (x'^0)^{-1} - V(x)\, x'^0, \tag{143}$$

where, according to definition (24),

$$x'^0 = \frac{\mathrm{d}x^0}{\mathrm{d}w} = \frac{\mathrm{d}t}{\mathrm{d}w}, \qquad x'^a = \frac{\mathrm{d}x^a}{\mathrm{d}w} . \tag{144}$$

The Lagrangian $\Lambda(x, x')$ is a *homogeneous function of first degree in the* x'^α. Now, we define a metric through the position [18]

$$g_{\alpha\beta}(x, x') = \frac{1}{2} \frac{\partial^2 \Lambda^2(x, x')}{\partial x'^\alpha \partial x'^\beta} . \tag{145}$$

Being $\Lambda^2(x, x')$ a homogeneous function of second degree in the x'^α, the $g_{\alpha\beta}(x, x')$ will be homogeneous functions of zero degree in the x'^α. The Euler theorem will give

$$\Lambda^2(x, x') = g_{\alpha\beta}(x, x')\, x'^\alpha x'^\beta. \tag{146}$$

Moreover, one defines $g^{\alpha\beta}(x, x')$ by means of

$$g_{\alpha\beta}\, g^{\gamma\beta} = \delta^{\gamma}{}_{\alpha}. \tag{147}$$

Also in the case of a natural conservative system, the $g_{\alpha\beta}$ have nonsimple expressions in terms of the old quantities:

$$\begin{cases} g_{00} = \dfrac{1}{2}\, \dfrac{\partial^2 \Lambda^2}{\partial^2 (x'^0)^2} = 3\mathscr{T}^2 + V^2, \\ g_{0a} = \dfrac{1}{2}\, \dfrac{\partial^2 \Lambda^2}{\partial x'^0 \partial x'^a} = -2\mathscr{T}\gamma_{ab}\, \dfrac{x'^b}{x'^0}, \\ g_{ab} = \gamma_{ac}\gamma_{bd}\, x'^c x'^d (x'^0)^{-2} + \gamma_{ab}(\mathscr{T} - V). \end{cases} \tag{148}$$

The Lagrange equations (29) of the system with Lagrangian (142) will coincide with the geodesic equations of the Finsler space with the metric (145). Moreover, if we choose s, the arclength defined by

$$\mathrm{d}s^2 = \Lambda^2(x, x')\, \mathrm{d}w^2, \tag{149}$$

as parameter (so that $\Lambda \equiv 1$), the geodesic equations will assume the simple form

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}s^2} + \gamma^\alpha{}_{\beta\gamma}\, \frac{\mathrm{d}x^\beta}{\mathrm{d}s}\, \frac{\mathrm{d}x^\gamma}{\mathrm{d}s} = 0, \tag{150}$$

where (see [18], chapt. I, II)

$$\gamma^\alpha{}_{\beta\gamma} = \gamma^\alpha{}_{\beta\gamma}(x, x') = \frac{1}{2}\, g^{\alpha\delta}\left(\frac{\partial g_{\delta\beta}}{\partial x^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta}\right). \tag{151}$$

7·2. *Generalized Killing equations in the Finsler space.* – Now, if we want to rewrite eqs. (35) (the so-called "generalized Killing equations") in the Finsler geometry with metric (145), we note that

$$\frac{\partial \Lambda}{\partial x^\alpha} = \frac{1}{2\Lambda}\, \frac{\partial \Lambda^2}{\partial x^\alpha} = \frac{1}{2\Lambda}\, \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\, x'^\beta x'^\gamma \tag{152}$$

and

$$\frac{\partial \Lambda}{\partial x'^\alpha} = \frac{1}{2\Lambda}\, \frac{\partial \Lambda^2}{\partial x'^\alpha} = \frac{1}{\Lambda}\, g_{\alpha\beta}\, x'^\beta + \frac{1}{2\Lambda}\, \frac{\partial g_{\beta\gamma}}{\partial x'^\alpha}\, x'^\beta x'^\gamma = \frac{1}{\Lambda}\, g_{\alpha\beta}(x, x')\, x'^\beta, \tag{153}$$

since the last term vanishes due to the Euler theorem. Thus,

$$p_\alpha = \frac{\partial \Lambda}{\partial x'^\alpha} = \frac{1}{\Lambda}\, g_{\alpha\beta}\, x'^\beta. \tag{154}$$

In place of eqs. (35), we get, by substitution:

$$\text{(155)}\qquad \begin{cases} \dfrac{1}{\Lambda} g_{\alpha\beta} x'^{\beta} \dfrac{\partial \xi^{\alpha}}{\partial x'^{\gamma}} = \dfrac{\partial f}{\partial x'^{\gamma}}, \\ x'^{\beta} x'^{\gamma} \left(\dfrac{\partial g_{\beta\gamma}}{\partial x^{\alpha}} \xi^{\alpha} + g_{\alpha\beta} \dfrac{\partial \xi^{\alpha}}{\partial x^{\gamma}} + g_{\alpha\gamma} \dfrac{\partial \xi^{\alpha}}{\partial x^{\beta}} \right) = 2\Lambda \dfrac{\partial f(x, x')}{\partial x^{\alpha}} x'^{\alpha}. \end{cases}$$

We consider these equations as the *generalized Killing equations* for the Finsler metric (145). In fact, if we have a space with a metric $g_{\alpha\beta}(x, x')$ and consider an infinitesimal point transformation

$$\text{(156)}\qquad \bar{x}^{\alpha} = x^{\alpha} + \varepsilon \xi^{\alpha}(x, x'),$$

generated by the vector $\boldsymbol{\xi}$, the necessary and sufficient condition in order that (156) is a motion (or, in other words, $\boldsymbol{\xi}$ is a Killing vector) is

$$\text{(157)}\qquad L_{\boldsymbol{\xi}} g_{\alpha\beta}(x, x') = 0,$$

where $L_{\mathbf{v}}$ denotes the Lie derivative along $\mathbf{v}$. Computing explicitly gives

$$\text{(158)}\qquad L_{\boldsymbol{\xi}} g_{\alpha\beta}(x, x') = \left(\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} \xi^{\gamma} + g_{\alpha\gamma} \frac{\partial \xi^{\gamma}}{\partial x^{\beta}} + g_{\gamma\beta} \frac{\partial \xi^{\gamma}}{\partial x^{\alpha}} \right) + \frac{\partial g_{\alpha\beta}}{\partial x'^{\gamma}} \left(\frac{\partial \xi^{\gamma}}{\partial x^{\delta}} x'^{\delta} + \frac{\partial \xi^{\gamma}}{\partial x'^{\delta}} x''^{\delta} \right).$$

Being $g_{\alpha\beta}$ and ξ^{α} functions of x^{β} and x'^{β} and not of x''^{β}, the condition (157) implies that the equations

$$\text{(159)}\qquad \begin{cases} \left(\dfrac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} \xi^{\gamma} + g_{\alpha\gamma} \dfrac{\partial \xi^{\gamma}}{\partial x^{\beta}} + g_{\gamma\beta} \dfrac{\partial \xi^{\gamma}}{\partial x^{\alpha}} \right) + \dfrac{\partial g_{\alpha\beta}}{\partial x'^{\gamma}} \dfrac{\partial \xi^{\gamma}}{\partial x^{\delta}} x'^{\delta} = 0, \\ \dfrac{\partial g_{\alpha\beta}}{\partial x'^{\gamma}} \dfrac{\partial \xi^{\gamma}}{\partial x'^{\delta}} = 0 \end{cases}$$

hold separately. It is easy to see that, if the vector $\boldsymbol{\xi}$ does not depend on the directions x'^{β}, the second equation of (159) reduces to an identity and the same will happen for the first one of (155) for $f \equiv 0$. In this case moreover, also the first equation in (159) and the second one in (155) will coincide (this will happen exactly if we formally retain the null term in the expression of eq. (153)). This justifies the use we have made of the term *generalized Killing equations*.

We can add further considerations to better understand the whole question. Coming back to the treatment of sect. **2**, we have that eq. (34) represents the condition such that the transformation generated by the vector field of components ξ^{α}, $\alpha = 0, 1, 2, \ldots, N$ be Noetherian in a generalized sense, that is the action integral (28) be invariant up to a gauge term under the transformation (31). If we now substitute into eq. (34) the equations of motion (29), the expression in the left-hand side results to be evaluated on the actual trajectories of the motion. If, then, we define the Finsler metric (145), (146), since the geodesics of such a metric coincide with the trajectories of the actual motion, by substituting into eq. (37) $\partial\Lambda/\partial x'^{\alpha}$ with $(1/\Lambda)\, g_{\alpha\beta} x'^{\beta} = p_{\alpha}$ we shall have that the obtained expression will result evaluated on the geodesics. Choosing then as parameter the arclength along the geodesics, we have, in place of eq. (34),

$$\frac{d}{ds}(p_\alpha)\,\xi^\alpha + p_\alpha \frac{d\xi^\alpha}{ds} - \frac{df}{ds} = 0\,. \tag{160}$$

The derivative with respect to s of a geometric object A, evaluated along the geodesics, is

$$\frac{dA}{ds} = A_{|\beta}\frac{dx^\beta}{ds}\,, \tag{161}$$

where the bar denotes the Cartan covariant derivative (see [18], chapt. II). Taking into account that

$$p_{\alpha|\beta} = 0\,, \qquad \frac{dx^\alpha}{ds_{|\beta}} = 0\,, \tag{162}$$

we shall obtain

$$p_\alpha\,\xi^\alpha{}_{|\beta}\frac{dx^\beta}{ds} - \frac{df}{ds} = 0\,. \tag{163}$$

But

$$p_\alpha\,\xi^\alpha{}_{|\beta}\frac{dx^\beta}{ds} = \frac{1}{\Lambda}g_{\alpha\gamma}\frac{dx^\gamma}{ds}\,\xi^\alpha{}_{|\beta}\frac{dx^\beta}{ds} = \xi_{\gamma|\beta}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} \tag{164}$$

($\Lambda = 1$ on the geodesics) and then, taking also into account the symmetry, we have finally

$$\frac{1}{2}(\xi_{\alpha|\beta} + \xi_{\beta|\alpha})\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} - \frac{df}{ds} = 0\,. \tag{165}$$

If we remember that the Lie derivative of the metric tensor in our case can also be written as

$$L_\xi g_{\alpha\beta}(x, x') = 2\xi_{(\alpha|\beta)} + 2\mathscr{C}_{\alpha\beta\gamma}\left[\left(\xi_{\gamma|\delta} + \frac{\partial\xi^\gamma}{\partial x'^\varrho}\mathscr{G}^\varrho{}_\delta\right)\frac{dx^\delta}{ds} + \frac{\partial\xi^\gamma}{\partial x'^\delta}\frac{d^2x^\delta}{ds^2}\right] + \\ + g_{\alpha\gamma}\frac{\partial\xi^\gamma}{\partial x'^\delta}\mathscr{G}^\delta{}_\beta + g_{\beta\gamma}\frac{\partial\xi^\gamma}{\partial x'^\delta}\mathscr{G}^\delta_\alpha\,, \tag{166}$$

where

$$\mathscr{C}_{\alpha\beta\gamma} = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x'^\gamma}\,, \qquad \mathscr{G}^\delta{}_\alpha = \frac{\partial\mathscr{G}^\delta}{\partial x'^\alpha} \tag{167}$$

and

$$2\mathscr{G}^\alpha = \gamma^\alpha{}_{\beta\gamma}x'^\beta x'^\gamma = \frac{1}{2}g^{\alpha\delta}\left[\frac{\partial g_{\delta\beta}}{\partial x^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta}\right]x'^\beta x'^\gamma\,, \tag{168}$$

one will have also

$$\xi_{(\alpha|\beta)} = \frac{1}{2} L_\xi g_{\alpha\beta}(x, x') - \mathscr{C}_{\alpha\beta\gamma}\left[\left(\xi_{\gamma|\delta} + \frac{\partial \xi^\gamma}{\partial x'^\varrho}\mathscr{G}^\varrho{}_\delta\right)\frac{dx^\delta}{ds} + \frac{\partial \xi^\gamma}{\partial x'^\delta}\frac{d^2 x^\delta}{ds^2}\right] - \\ - 2 g_{\gamma(\alpha}\mathscr{G}^\delta{}_{\beta)}\frac{\partial \xi^\gamma}{\partial x'^\delta} . \tag{169}$$

Since, due to the homogeneity properties of the $g_{\alpha\beta}$, one has

$$\mathscr{C}_{\alpha\beta\gamma}\frac{dx^\alpha}{ds} = \mathscr{C}_{\alpha\beta\gamma}\frac{dx^\beta}{ds} = \mathscr{C}_{\alpha\beta\gamma}\frac{dx^\gamma}{ds} = 0 , \tag{170}$$

we shall have finally

$$\frac{1}{2}\xi_{(\alpha|\beta)}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} - \frac{df}{ds} = \frac{1}{4} L_\xi g_{\alpha\beta}(x, x')\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} - \\ - g_{\gamma(\alpha}\mathscr{G}^\delta{}_{\beta)}\frac{\partial \xi^\gamma}{\partial x'^\delta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} - \frac{df}{ds} = 0 . \tag{171}$$

When $\xi^\alpha = \xi^\alpha(x)$, this reduces to

$$\frac{1}{4} L_\xi g_{\alpha\beta}(x, x')\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} - \frac{df}{ds} = 0 . \tag{172}$$

Therefore, in this case, if $\boldsymbol{\xi}$ is a Killing vector ($L_\xi g_{\alpha\beta} = 0$), the problem is solved by $f \equiv 0$. When $\xi^\alpha = \xi^\alpha(x, x')$, the problem cannot be solved by a Killing vector: we need a *generalized Killing vector*, that is a vector which satisfies the complete equation with $f \neq 0$. Obviously, there may exist also the possibility that the term

$$g_{\gamma(\alpha}\mathscr{G}^\delta{}_{\beta)}\frac{\partial \xi^\gamma}{\partial x'^\delta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} \tag{173}$$

vanishes even if the derivatives $\partial\xi^\gamma/\partial x'^\delta$ are not individually zero. In this case the solution of eq. (171) is again a true Killing vector.

7·3. *Examples of generalized Killing vectors.* – In order to clarify the implications of the formalism we have set up in the previous section, we will revisit one of the classical examples already examined. In fact we shall limit ourselves to the problem of the Kepler motion, not to make our exposition excessively dull. Anyhow, it is immediate to apply the same procedure in the other cases. The Lagrangian will be given by

$$\Lambda(x, x') = \frac{1}{2}[(x'^1)^2 + (x'^2)^2](x'^0)^{-1} + \frac{\mu}{r} x'^0 \tag{174}$$

and the coefficients of the Finsler metric of the relevant three-dimensional space by

$$\left\{\begin{aligned} g_{00} &= \frac{3}{2}[(x'^1)^2 + (x'^2)^2](x'^0)^{-2} + \frac{\mu^2}{r^2}, \\ g_{11} &= (x'^1)^2(x'^0)^{-2} + \Lambda(x, x')(x'^0)^{-1}, \\ g_{22} &= (x'^2)^2(x'^0)^{-2} + \Lambda(x, x')(x'^0)^{-1}. \end{aligned}\right. \tag{175}$$

7·3.1. Energy integral. If, as in eqs. (155), we consider the transformation given by

$$\xi^0 = -1, \qquad \xi^i \equiv 0, \qquad f \equiv 0, \tag{176}$$

from eqs. (38) and (40) we have immediately

$$I = \mathscr{H}. \tag{177}$$

On the other hand, from the transformation

$$\xi^0 = 0, \qquad \xi^i = x'^i, \tag{178}$$

we have from the first of equations (155):

$$\frac{1}{\Lambda} g_{\alpha\beta} x'^\beta x'^\beta \delta^{\alpha\gamma} = \frac{\partial f}{\partial x'^\gamma}, \tag{179}$$

which gives $f(x, x') = \Lambda(x, x')$. The last of equations (155) then becomes

$$\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \xi^\alpha x'^\alpha x'^\beta x'^\gamma = 2\Lambda \frac{\partial \Lambda}{\partial x^\alpha} x'^\alpha = \frac{\partial \Lambda^2}{\partial x^\alpha} = \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \xi^\alpha x'^\alpha x'^\beta x'^\gamma \tag{180}$$

and is therefore identically satisfied. The first integral corresponding to the transformation generated by (178) will be

$$I = \frac{\partial \Lambda}{\partial x'^\alpha} \xi^\alpha - f = \frac{\partial \Lambda}{\partial x'^i} \xi^i - f = p_i x'^i - \Lambda = (p_i \dot{x}^i - \mathscr{L}) \frac{\mathrm{d}t}{\mathrm{d}w}. \tag{181}$$

Taking $t = w$ this gives $I = \mathscr{H}$.

7·3.2. Angular-momentum integral. The transformation given by

$$\xi^0 \equiv 0, \qquad \xi^1 = -x^2, \qquad \xi^2 = x^1, \qquad f \equiv 0, \tag{182}$$

corresponds to the angular-momentum integral

$$I = x^1 p_2 - x^2 p_1. \tag{183}$$

7·3.3. The Laplace-Runge-Lenz vector. Let us consider now again the two-dimensional Kepler problem and a transformation generated by

$$\xi^0 = 0, \qquad \xi^a = A^a{}_{\beta\gamma} x'^\beta x^\gamma x'^0, \qquad a = 1, 2 \tag{184}$$

with the coefficients $A^a{}_{\beta\gamma}$ given by constants. The first of equations (155) will become

$$p_\alpha \frac{\partial \xi^\alpha}{\partial x'^\gamma} = \frac{\partial f}{\partial x'^\gamma}\,. \tag{185}$$

Taking into account that the Lagrangian is given by

$$\Lambda(x, x') = \frac{1}{2}[(x'^1)^2 + (x'^2)^2](x'^0)^{-1} + \frac{\mu}{r} x'^0, \tag{186}$$

we obtain that one solution for eq. (185) and the second of equations (155) is given by

$$\begin{cases} f(x, x') = A\left[-x^1(x'^2)^2 + x^2 x'^1 x'^2 - \mu \dfrac{x^1}{r}(x'^0)^2\right], \\ A^1{}_{11} = A^1{}_{12} = A^2{}_{11} = 0, \qquad A^2{}_{12} = A\,, \\ A^1{}_{21} = A^2{}_{22} = 0, \qquad A^2{}_{21} = -2A, \qquad A^1{}_{22} = A\,. \end{cases} \tag{187}$$

By choosing $A = -1/\mu$, we get

$$\begin{cases} \xi^1 = Ax^2 x'^2 x'^0 = Ax^2 \dot{x}^2 (x'^0)^2 = -\dfrac{1}{\mu} x^2 \dot{x}^2 (x'^0)^2, \\ \xi^2 = A[x^2 x'^1 x'^0 - 2x^1 x'^2 x'^0] = -\dfrac{1}{\mu}(\dot{x}^1 x^2 - 2x^1 \dot{x}^2)(x'^0)^2, \end{cases} \tag{188}$$

so that

$$I = -A[x^1(x'^2)^2 - x^2 x'^1 x'^2 - \mu \frac{x^1}{r}(x'^0)^2] = \tag{189}$$

$$= \frac{1}{\mu}[x^1(x'^2)^2 - x^2 x'^1 x'^2 - \mu \frac{x^1}{r}](x'^0)^2.$$

With $w = t$, it results that $I = I^{(1)}_{\mathrm{LRL}}$ (see eq. (94) above). The other solution for eq. (185) and the second of equations (155), analogously, will result in $I = I^{(2)}_{\mathrm{LRL}}$.

7·4. *Remark.* – If we refer to the remark of subsect. **6**·5 and to the conclusion of subsect. **7**·2, we realize that it is possible to make the integral of eq. (189) correspond to a Killing vector (not a generalized one), defining the time component of this vector as

$$\xi^0 = \frac{1}{\mu} \frac{[-x^1(x'^2)^2 + x^2 x'^1 x'^2 - \mu(x^1/r)(x'^0)^2]}{\Lambda(x, x')(x'^0)^{-1}} \tag{190}$$

and its spatial components as

$$\xi^a = \bar{\xi}^a + \xi^0 x'^a (x'^0)^{-1}, \tag{191}$$

being eq. (191), eq. (21) rewritten in homogeneous variables and $\bar{\xi}^a$ standing for the ξ^a in eq. (184). The same can be obtained, obviously, for the integral corresponding to $I^{(2)}_{\mathrm{LRL}}$. Therefore, at the price of a considerable complication of the expressions of the components of the generator, we can obtain that the conserved Laplace-Runge-Lenz vector corresponds to a Killing vector of the Finsler metric (175).

8. – Conclusions

As we have seen, given a dynamical system subjected to monogenic forces and then represented in homogeneous variables by a Lagrangian function $\Lambda(x, x')$ in which, in general, also the time coordinate $t = x^0$ can appear, it is always possible "to geometrize" it completely, *i.e.* to incorporate the generalized work function into the metric. In the examples we have considered, the motion of a particle in given potentials, one obtained the motion of a free particle in a curved space (Finsler manifold) whose metric tensor was depending also on the direction (velocity). The trajectories allowed in the actual motion thus became the geodesics of this curved space. It is known that it is possible to derive from a Lagrangian also particular dissipative forces depending only on the velocities and, in case, on the time [19, 20] by means of a suitable transformation of the independent variable.

We can then conclude that, for a large part of the systems considered in the applications, it is possible to obtain the geometrization in a Finsler space. For a narrower class, the conservative systems at total fixed energy, the geometrization results much simpler: one has the so-called Jacobi geometry with Riemannian metric. In the cases we have so far recalled, the first integrals of the dynamical system come to coincide, or to almost coincide, with the first integrals of the geodesic equations. Let us analyze the first case. This happens when the generators of the Noetherian transformations do not depend on the velocities:

$$\xi^\alpha = \xi^\alpha(x), \qquad \alpha = 1, 2, \ldots, N. \tag{192}$$

In this case the generalized Killing equations have always solutions with $f \equiv 0$ and therefore we are in the presence of Killing vectors and not of generalized Killing vectors; the Noetherian symmetries are true symmetries and the first integrals are linear in the momenta. The second case happens when the generators depend also on the velocities (we limit ourselves to consider only first derivatives—for higher derivatives see Olver [14]):

$$\xi^\alpha = \xi^\alpha(x, x'), \qquad \alpha = 1, 2, \ldots, N. \tag{193}$$

Now the generalized Killing equations cannot in general be solved by $f = 0$ and the Noetherian symmetries are "quasi-symmetries"; the inner product between a Killing vector and the tangent vector to a geodesic is no more constant on the geodesic itself [21]: to obtain this we must resort to a generalized Killing vector ($f \neq 0$). The corresponding first integrals will be at least quadratic in the momenta.

In the Jacobi geometry, as we have seen, this case corresponds to the introduction of the Killing tensors. Since, as we have recalled in sect. **2**, to a first integral many Noetherian transformations can correspond, to the same integral both symmetries and quasi-symmetries can correspond (the example we have reported regards the Kepler problem). The geometrization in a Finsler space of the system characterized, in homogeneous variables, by a Lagrangian $\Lambda(x, x')$ is the most general and clear way to make correspond to the Noetherian symmetries and quasi-symmetries rigorous geometrical concepts. It is evident that in the first case (Killing vectors) the existence of theorems already well established provides us with a certain amount of knowledge and tools for the research (maximum number of Killing vectors admitted by a given space, integrability conditions for the Killing equations, etc.). In the second case (generalized Killing vectors), it does not exist at present an already established geometrical theory. From the examples given above, one could be induced to think that this last case never occurs in mechanics and that, therefore, in the Finsler spaces related to the mechanical problems we have only symmetries. However, it can be checked that there exist several dynamical systems in which quasi-symmetries are present (see, *e.g.*, the Emden-Fowler equation in Dyukic [4]).

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