Gravitoelectromagnetism in rotating black-hole space-times (*)

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Summary. — Gravitoelectromagnetism in the Kerr space-time is studied in analogy with electromagnetism in the Kerr-Newman space-time. A recent suggestion by Semerak that there is a close similarity between the Kerr-Newman electromagnetic fields and the Kerr gravitoelectromagnetic ones is reanalysed and clarified as referring to the fields measured by the static observers in the two geometries.

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1. – Introduction

Over the years the analogy between the electromagnetic field equations and the gravitational field equations in the linearized approximation to general relativity has drawn much interest, see, *e.g.*, Forward [1] and Thorne [2-5]. While various ways of establishing a nonlinear analogy had also been introduced independently of one another for strong gravitational fields, only more recently has a very powerful and systematic approach been taken in identifying and relating to each other the various ways in which the gravitoelectromagnetic field can be described by splitting the fully nonlinear gravitational field in the strong field regime [6].

In a recent paper, Semerák [7] has noticed a close similarity between the electromagnetic lines of force in the Kerr-Newman space-time and the corresponding gravitoelectromagnetic counterparts in the Kerr space-time, both measured by the zero-angular-momentum observers (ZAMOs), also known as the local nonrotating observers (LNROs), of each geometry in the strong field limit, when an obvious

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correspondence is made between the two space-time manifolds using the Boyer-Lindquist coordinates. However, Semerak's definition of electromagnetic fields is not directly related to their measurement by any family of observers, so the problem requires further investigation.

Here we study this question in the general framework of gravitoelectromagnetism, a powerful formalism for studying the local splitting of a space-time by any family of observers. Examining the values of the standard definitions of splitting fields (electromagnetic and gravitoelectromagnetic) as measured by a congruence of observers [6], we find that the correspondence noticed by Semerák does exist for the static observers which follow the time lines in the Boyer-Lindquist coordinates but for the ZAMO only the Kerr-Newman magnetic field lines reduce to the corresponding Kerr gravitomagnetic ones. These coincidences are special and do not hold for general families of observers.

Briefly, gravitoelectromagnetism is the description of splitting space-times into space plus time. The common feature of space-time splittings is the introduction of a family of observers and an accompanying measurement process by which tensors and tensor equations are orthogonally decomposed with respect to the observer four-velocity field, reflecting the local splitting of each tangent space into the local rest space and the local time direction of the observer. With the additional structure of a slicing of the space-time threaded by a family of transversal curves adapted to the observer four-velocity, additional splitting fields arise. For example, the spatial metrics arises from the spatial projection of the space-time metric tensor, while the lapse and shift arise from the splitting of the differential of a slicing time coordinate in the "threading point of view" or of the generating vector field of the threading congruence in the "slicing point of view", where the observer four-velocity is taken to be the unit timelike vector field along the threading (m) or normal to the slicing (n), respectively.

For stationary axisymmetric space-times, a natural slicing and threading of the space-time exists determined by the geometry of the space-time itself, leading to two natural families of preferred observer families. In the case of a Kerr-Newman black hole, the preferred threading observers are the static observers, while the preferred slicing observers are the locally nonrotating observers.

2. – The metric and electromagnetic-field space-time quantities

That the most general asymptotically flat solution of the stationary axially symmetric Einstein-Maxwell equations endowed with angular momentum is given by the Kerr-Newman solution was proved by Carter [8]. In Boyer-Lindquist coordinates the form of the metric is the following [9]:

$$\mathrm{d}s^2 = -\left(1 - \frac{V}{\varrho^2}\right)\mathrm{d}t^2 - \frac{2aV\sin^2\theta}{\varrho^2}\,\mathrm{d}t\,\mathrm{d}\phi + \frac{\varrho^2}{\Delta}\,\mathrm{d}r^2 + \varrho^2\,\mathrm{d}\theta^2 + \frac{\Lambda\sin^2\theta}{\varrho^2}\,\mathrm{d}\phi^2, \tag{2.1}$$

where

$$\begin{aligned} &\varrho^2 = r^2 + a^2\cos^2\theta\,, && V = 2mr - q^2,\\ &\Delta = r^2 + a^2 - V\,, && \Lambda = -\Delta a^2\sin^2\theta + (r^2+a^2)^2 = \Delta\varrho^2 + V(r^2+a^2)\,,\\ & && \Lambda = (r^2+a^2)\,\varrho^2 + Va^2\sin^2\theta\,, \end{aligned}$$

and $\sqrt{^{(4)}g} = \varrho^2 \sin\theta$. The electromagnetic four-potential is

$$(2.2)\qquad {}^{(4)}A^{\flat} = -\frac{qr}{\varrho^2}(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi)\,, \qquad {}^{(4)}A^{\natural} = \frac{qr(r^2+a^2)}{\Delta\varrho^2}\left[\partial_t + \frac{a}{r^2+a^2}\partial_\phi\right],$$

where the two symbols $\flat$ and $\natural$ denote, respectively, the totally covariant and contravariant form of a tensor, and the associated elecromagnetic field is

$$(2.3)\qquad {}^{(4)}F \equiv \frac{1}{2}\,{}^{(4)}F_{\alpha\beta}\,\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta =$$

$$= \frac{q}{\varrho^4}(r^2 - a^2\cos^2\theta)\,\mathrm{d}r \wedge (\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi) + 2\,\frac{q}{\varrho^4}\,ar\sin\theta\cos\theta\,\mathrm{d}\theta \wedge$$

$$\wedge [(r^2+a^2)\,\mathrm{d}\phi - a\,\mathrm{d}t]\,.$$

It has been shown (see [10-14]) that the electromagnetic lines of force are a very powerful tool for understanding the astrophysical applications of the electric and magnetic structures of a black hole. Using an orthonormal tetrad adapted to the family of locally nonrotating observers, *i.e.* with timelike member

$$\omega^{\hat{0}} = u^{\flat}_{\mathrm{LNRO}} = -\sqrt{\Delta\varrho^2/\Lambda}\,\mathrm{d}t\,,$$

Ruffini [11] studied the electric and magnetic fields and their corresponding lines of force. In the same paper, Ruffini also studied the electric and magnetic fields measured by the family of "Carter's observers" which are adapted to the electromagnetic structure of the solution, *i.e.* whose four-velocity field

$$(2.4)\qquad u^{\flat}_{\mathrm{car}} = \sqrt{\frac{\Delta}{\varrho^2}}(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi)\,, \qquad u^{\natural}_{\mathrm{car}} = -\frac{1}{\sqrt{\Delta\varrho^2}}[(r^2+a^2)\,\partial_t + a\partial_\phi]\,,$$

is proportional to the electromagnetic four-potential

$$(2.5)\qquad {}^{(4)}A^{\flat} = -\frac{qr}{\sqrt{\Delta\varrho^2}}\,u^{\flat}_{\mathrm{car}}\,.$$

Here we consider static observers whose four-velocity is proportional to the timelike Killing vector (aligned with the Boyer-Lindquist time coordinate lines)

$$(2.6)\qquad u_{(\mathrm{stat})} = \frac{1}{\sqrt{1 - V/\varrho^2}}\,\partial_t.$$

All these observer families—the locally nonrotating ones, the static ones, and Carter's observers—are natural for studying the charged Kerr black hole. In fact the first two come out directly from the geometry, *i.e.* the gravitational field, while the third one is connected (adapted) to the electromagnetic field.

3. – Splitting the Kerr-Newman space-time variables

The slicing by the family of time coordinate hypersurfaces and the threading by the time coordinate lines of the Boyer-Lindquist coordinates leads to the following representation of the threading-generating vector field and the slicing 1-form, respectively, in the slicing and threading points of view:

$$\partial_t = Nn + N^\phi \partial_\phi , \qquad dt = M^{-1}(-m^\flat) + M_\phi d\phi . \tag{3.1}$$

The lapse and shift fields in the two points of view act as potentials for these respective spatial gauge fields used to solder the observer tangent space splitting to the manifold structure of the space-time through a choice of coordinate system.

Following the notational conventions of Jantzen, Carini, and Bini [6],

$$ds^2 = \begin{cases} -M^2(dt - M_\phi d\phi)^2 + \gamma_{ab} dx^a dx^b & \text{(threading)} , \\ -N^2 dt^2 + g_{ab}(dx^a + N^a dt)(dx^b + N^b dt) & \text{(slicing)} , \end{cases} \tag{3.2}$$

with ($x^1 = r$, $x^2 = \theta$, $x^3 = \phi$), the space-time splitting associated with the threading and slicing observers produces the following spatial gravitational potentials and fields (see table I).

Let us confine our attention to the threading (static) observers with four-velocity $m = u_{(\mathrm{stat})}$; the gravitoelectromagnetic fields associated with them are

$$\begin{cases} g(m)^\natural \equiv -\mathrm{grad}_m(\ln M) - \mathcal{L}(m)_{e_0}\mathbf{M} = \\ \qquad = -\dfrac{1}{\varrho^6 M^2}[\Delta(Vr - m\varrho^2)\,\partial_r - a^2 V \sin\theta\cos\theta\partial_\theta] , \\ H(m)^\natural \equiv M\,\mathrm{curl}_m \mathbf{M} = -\dfrac{2a}{\varrho^6 M^2}[V\Delta\cos\theta\partial_r + \sin\theta(Vr - m\varrho^2)\,\partial_\theta] , \end{cases} \tag{3.3}$$

Table I. – *Splitting definitions.*

	Threading (outside the ergosurface)	Slicing (outside the horizon)
lapse	$M^2 = 1 - \dfrac{V}{\varrho^2}$	$N^2 = \dfrac{\Delta\varrho^2}{\Lambda}$
shift	$M_\phi = -\dfrac{aV\sin^2\theta}{\varrho^2 - V}$	$N_\phi = -\dfrac{Va}{\varrho^2}\sin^2\theta$
spatial metric	$\gamma_{rr} = {}^{(4)}g_{rr}$ $\gamma_{\theta\theta} = {}^{(4)}g_{\theta\theta}$ $\gamma_{\phi\phi} = \dfrac{\Delta\varrho^2\sin^2\theta}{\varrho^2 - V}$ $\sqrt{\gamma} = \dfrac{\varrho^2\sin\theta}{M}$	$g_{rr} = \gamma_{rr} = {}^{(4)}g_{rr}$ $g_{\theta\theta} = \gamma_{\theta\theta} = {}^{(4)}g_{\theta\theta}$ $g_{\phi\phi} = {}^{(4)}g_{\phi\phi}$ $\sqrt{g} = \dfrac{\varrho^2\sin\theta}{N}$

while in this point of view the gravitomagnetic symmetric tensor field is identically zero,

$$\theta(m)^\flat = \frac{1}{2}\mathscr{L}(m)_m{}^{(4)}g^\flat \equiv 0\,. \tag{3.4}$$

These fields [6] determine the threading (spatial) gravitational force $F^{(G)}_{(\text{tem})}(U, m)$, acting on the test particle with four-velocity $U = \gamma(U, m)[u + \nu(U, m)]$,

$$F^{(G)}_{(\text{tem})}(U, m) = m\gamma(U, m)[g(m) + \varepsilon_{(\text{tem})}\nu(U, m)\times_m H(m) + \sigma_{(\text{tem})}\theta(m) \,\mathrm{L}\, \nu(U, m)]\,, \tag{3.5}$$

with

$$\varepsilon_{(\text{tem})} = \left(\frac{1}{2}, 1, 1, 1\right), \qquad \sigma_{(\text{tem})} = (-1, -1, -2, 0), \qquad \text{tem} = \text{fw}, \text{cfw}, \text{lie}, \text{lie}\flat\,,$$

in analogy with the electromagnetic Lorentz force $F^{(\text{EM})}(m)$,

$$F^{(\text{EM})}(m) = q[E(m) + \nu(U, m)\times_m B(m)]\,, \tag{3.6}$$

where (tem) is a convenient notation for handling the various ways (Fermi-Walker, corotating Fermi-Walker and Lie) in which the observer family (m) can transport its spatial frames along its worldlines (see [6], sect. **6** for a detailed derivation of these formulas) and where the electric and magnetic fields arise from the standard splitting of the electromagnetic 2-form

$${}^{(4)}F \equiv d\,{}^{(4)}A = m^\flat \wedge E(m) + {}^{*(m)}B(m) \tag{3.7}$$

yielding explicitly

$$\begin{cases} E(m)^\natural = \dfrac{q}{\varrho^6 M}[\Delta(r^2 - a^2\cos^2\theta)\,\partial_r - 2a^2 r\sin\theta\cos\theta\partial_\theta]\,,\\[2mm] B(m)^\natural = \dfrac{aq}{\varrho^6 M}[2r\Delta\cos\theta\partial_r + \sin\theta(r^2 - a^2\cos^2\theta)\,\partial_\theta]\,. \end{cases} \tag{3.8}$$

4. – Lines of force

A very powerful tool for visualizing electric and magnetic fields is in the study of their lines of force. Here we do this in a qualitative way for the threading electromagnetic (see fig. 1 and 2) and gravitoelectromagnetic (fig. 3 and 4) fields in the Kerr-Newman space-time. A more complete picture (*i.e.* not only qualitative) of these fields can be given taking into account the flux through a surface whose contour is a ring at fixed r and θ in such a way that the number of lines drawn (representative of the intensity of the field) would be proportional to the flux itself. We will discuss this in a

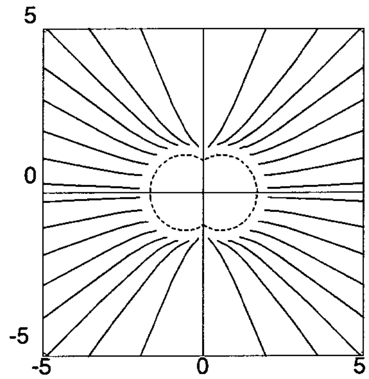

Fig. 1. – Kerr-Newman threading electric lines of force. For all figures the choice of metric parameters is $m = 1$, $a = q = 1/\sqrt{2}$ and the two axes are the Cartesian-like coordinates (x, y) related to the (spherical at spatial infinity) Boyer-Lindquist pair (r, θ) by $x = r\sin\theta$ and $y = r\cos\theta$. Note that in each of these figures the distribution of the selected lines of force is only qualitative and does not model the field intensity.

subsequent paper. However, the Kerr-Newman threading electric lines of force (fig. 1) satisfy

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{\Delta(r^2 - a^2\cos^2\theta)}{2a^2 r\sin\theta\cos\theta} \tag{4.1}$$

and are orthogonal to the surfaces $A_0(r, \theta) = \text{const}$, while for the magnetic lines of force (fig. 2) we have

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{2r\Delta\cos\theta}{\sin\theta(r^2 - a^2\cos^2\theta)} . \tag{4.2}$$

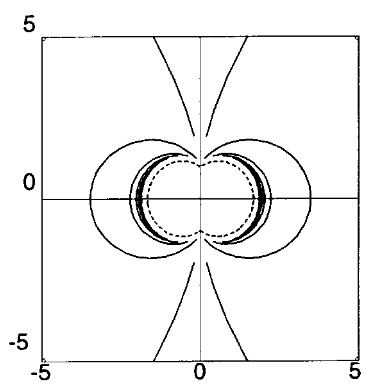

Fig. 2. – Threading magnetic lines of force.

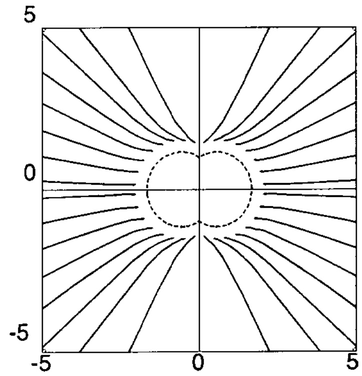

Fig. 3. – Threading gravitoelectric lines of force.

Figure 3 shows instead the gravitoelectric lines of force which satisfy

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{\Delta(Vr - m\varrho^2)}{a^2 V \sin\theta\cos\theta} \tag{4.3}$$

and are orthogonal to the surfaces $M = \text{const}$, while in fig. 4 are shown the gravitomagnetic ones satisfying

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{V\Delta\cos\theta}{\sin\theta(Vr - m\varrho^2)} \tag{4.4}$$

and coinciding with the lines $M_\phi = \text{const}$.

In each figure the choice of parameters $m = 1$, $a = q = 1/\sqrt{2}$ has been adopted. It is easy to see that the limit $q \to 0$ of the threading electromagnetic lines of force is nontrivial (even though each of the electromagnetic fields in the same limit vanishes) and coincides with the lines of force in the Kerr black hole as measured by the same threading observers for that geometry. This clarifies the ambiguous remark made by

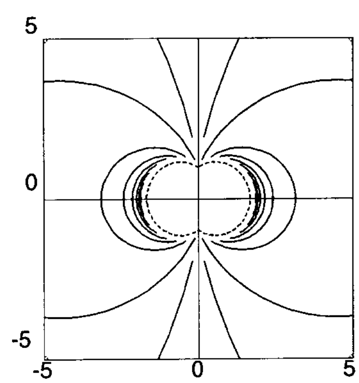

Fig. 4. – Threading gravitomagnetic lines of force.

Semerák. In particular, the following limits hold for the fields themselves:

$$\lim_{q\to 0}\left(\frac{E(m)}{q}\,,\;\frac{B(m)}{q}\right)\Bigg|_{\text{Kerr-Newman}}=\left(-\frac{g(m)\,M}{m}\,,\;-\frac{H(m)\,M}{2m}\right)\Bigg|_{\text{Kerr}}, \tag{4.5}$$

as one can easily see from the previous formulas (3.9), (3.8).

5. – Conclusions

We have shown that there exists an exact correspondence between the Kerr-Newman electromagnetic fields and the Kerr gravitoelectromagnetic ones as measured by the threading observers in the two distinct geometries, clarifying what was first noticed by Semerak. It is of interest to further analyse the consequences of this correspondence by studying comparable test particle motion as described by the threading observers in the two distinct geometries.

Finally, it is easy to understand that this result cannot be valid for general observer families. However, the slicing observers (LNROs) are so special that this correspondence is not completely lost. In a forthcoming paper [15] we will show in detail how this correspondence is broken by the slicing electric-gravitoelectric fields but still holds for the magnetic-gravitomagnetic ones, and we will apply all of these results to the study of test particle motion.

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