Scattering of spinning test particles by gravitational plane waves (*)

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Summary. — We study the motion of spinning particles in the gravitational plane-wave background and discuss particular solutions under a suitable choice of supplementary conditions. An analysis of the discontinuity of the motion across the wavefront is presented too.

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1. – Introduction

In general relativity a small body can be identified with a point particle, governed by the geodesic equation of motion, only in first approximation, *i.e.* if its inner structure is completely neglected. Multipole particles provide us with a higher-order approximation of the whole extended body. Multipole expansion for test particles can be introduced in different ways: by considering a point particle which is supposed to give a small disturbance to the background metric [1]; by considering a singular worldline which is support of a distribution-valued matter-energy tensor [2]; or by considering a small world tube $\mathscr{T}$, support of an ordinary matter-energy tensor $T^{\alpha\beta}$ (see, for example, [3-6]). The last method is the most used, and to this method we refer in the following.

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1$\cdot$1. *Equations of motion of the spinning particle.* – Let the cuvature tensor be defined as follows:

$$R_{\alpha\beta\varrho}{}^{\sigma} \stackrel{\text{def}}{=} 2\partial_{[\beta}\Gamma_{\alpha]\varrho}{}^{\sigma} + 2\Gamma_{\varrho[\alpha}{}^{\nu}\Gamma_{\beta]\nu}{}^{\sigma}, \tag{1}$$

where $\Gamma_{\alpha\beta}{}^{\sigma}$ are the Christoffel symbols.

The equations of motion of a spinning particle (also called "dipole" particle, which is the first step of the multipole expansion after the point particle or single-pole), have been derived by Papapetrou [3]:

$$\frac{\mathrm{D}}{\mathrm{d}\tau_U} P^{\alpha} = \frac{1}{2} R_{\varrho\sigma\beta}{}^{\alpha} S^{\varrho\sigma} U^{\beta}, \qquad \frac{\mathrm{D}}{\mathrm{d}\tau_U} S^{\alpha\beta} = P^{\alpha} U^{\beta} - P^{\beta} U^{\alpha}, \tag{2}$$

where

$$P^{\alpha} \stackrel{\text{def}}{=} mU^{\alpha} + U_{\beta} \frac{\mathrm{D}}{\mathrm{d}\tau_U} S^{\beta\alpha}$$

is the (generalized) momentum vector, $S^{\alpha\beta}$ is an antisymmetric "spin" tensor, $m = -\mathbf{P}\cdot\mathbf{U}$ is the mass of the particle, and $\mathbf{U}$ is the tangent unit vector of the "center line" of the multipole reduction, arbitrarly chosen between the generatrices of $\mathscr{T}$. Units are chosen in order to have the speed of light in empty space $c = 1$. Here $\mathrm{D}/\mathrm{d}\tau_U = \nabla_U$ means covariant derivative along $\mathbf{U}$ (eventually, if the support of the tensor to be derived is the center line only, this operation can be defined with the help of an arbitrary prolongation to an open domain).

These are 7 independent equations for 10 unknown quantities: 3 further scalar equations (supplementary conditions) are needed for the scheme to be completed.

1$\cdot$2. *Supplementary conditions.* – Usually, supplementary conditions are of two kinds: constitutive equations or "background conditions". Constitutive equations are intended to describe the inner structure of the considered particle; background conditions are usually introduced with the further intent of simplifying the equations of motion when studied in a particular background. In any case, the typical condition is a simmetry condition for the spin tensor with respect to a direction $\mathbf{V}$:

$$S^{\alpha\beta} V_{\beta} = 0\ . \tag{3}$$

If $\mathbf{V}$ is an element of the particle motion, (tipically $\mathbf{U}$ or $\mathbf{P}$), this is a constitutive equation; if $\mathbf{V}$ is suggested by the background, it is a background condition.

The two alternative constitutive equations are

$$S^{\alpha\beta} U_{\beta} = 0\ , \qquad S^{\alpha\beta} P_{\beta} = 0\ . \tag{4}$$

Condition $(4)_1$ [1, 7, 2] has a very interesting physical interpretation, as was recently pointed out in [8]. The main problem with this hypothesis is that equations of motion involve second derivatives, and thus the solution depends on the initial acceleration. This case is similar to that of the radiating charged point particle in an electromagnetic field ([9]): non-physical motions are allowed.

Condition $(4)_2$ avoids this problem but seems to be less physically significant, also because **P** must in addition be supposed time-like for the motion to be determined [10].

Both constitutive equations have been examined in the framework of the spatial tensor algebra of the local rest space of the particle by Ferrarese *et al.* (see [11]).

Background conditions ([12, 13] p. 336) are less interesting from the theoretical point of view, but they are useful for handling the equations of motion in particular cases.

The motion in given backgrounds has been studied in the cases of the following solutions: Schwarzschild [12], Kerr [14], and Vaidya ([13] p. 336). Here we are going to examine the case of the gravitational plane-wave background ([15] p. 957).

2. – Motion in the wave background

The gravitational plane-wave line element (see [15]) can be written as follows:

$$ds^2 = -2\,\mathrm{d}u\,\mathrm{d}v + F^2(u)\,\mathrm{d}x^2 + G^2(u)\,\mathrm{d}y^2, \qquad u = \frac{t-z}{\sqrt{2}}, \qquad v = \frac{t+z}{\sqrt{2}} \tag{5}$$

in terms of the two null coordinates u and v.

The metric (5) gives rise to the following non-null Christoffel symbols

$$\Gamma_{xx}{}^{v} = FF', \qquad \Gamma_{yy}{}^{v} = GG', \qquad \Gamma_{ux}{}^{x} = \frac{F'}{F}, \qquad \Gamma_{uy}{}^{y} = \frac{G'}{G} \tag{6}$$

with $(\ )' = \mathrm{d}/\mathrm{d}u$. We can also write

$$\Gamma_{\alpha\beta}{}^{\sigma} = (FF'\,\delta_{\alpha x}\delta_{\beta x} + GG'\,\delta_{\alpha y}\delta_{\beta y})\,\delta_{\sigma v} + 2\,\frac{F'}{F}\,\delta_{u(\alpha}\delta_{\beta)x}\delta_{\sigma x} + 2\,\frac{G'}{G}\,\delta_{u(\alpha}\delta_{\beta)y}\delta_{\sigma y}. \tag{7}$$

The resulting compact expression of the curvature tensor is

$$R_{\alpha\beta\varrho\sigma} = 4\,\frac{F''}{F^3}\,g_{x[\alpha}g_{\beta]v}g_{v[\varrho}g_{\sigma]x} + 4\,\frac{G''}{G^3}\,g_{y[\alpha}g_{\beta]v}g_{v[\varrho}g_{\sigma]y}. \tag{8}$$

The Riemann tensor also admits the following simple 6-dimensional expression, in terms of our skew-symmetric null coordinates 2-forms:

$$R_{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -F''/F^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -G''/G^3 \end{pmatrix}, \tag{9}$$

where $A, B = \partial_u \wedge \partial_v,\ \partial_u \wedge \partial_x,\ \partial_u \wedge \partial_y,\ \partial_x \wedge \partial_y,\ \partial_v \wedge \partial_x,\ \partial_v \wedge \partial_y$, and where we have defined $(\partial_\alpha \wedge \partial_\beta)_{\mu\nu} \overset{\text{def}}{=} 2 g_{\alpha[\mu} g_{\nu]\beta}$.

For what concerns the Ricci tensor $R_{\beta\varrho} \overset{\text{def}}{=} R_{\alpha\beta\varrho}{}^{\alpha}$, we can get, from (8),

$$R_{\beta\varrho} = \left(\frac{F''}{F} + \frac{G''}{G} \right) g_{v\beta} g_{v\varrho} \tag{10}$$

and, consequently, the Einstein equations *in vacuo* reduce to

$$\frac{F''}{F} + \frac{G''}{G} = 0 . \tag{11}$$

The equations of the geodesic curves, in terms of the unitary tangent vector **U**, are

$$\begin{cases} \dot{U}^u = 0 , \\ \dot{U}^v + (U^x)^2 FF' + (U^y)^2 GG' = 0 , \\ \dot{U}^x + 2 \dfrac{F'}{F} U^u U^x = 0 , \\ \dot{U}^y + 2 \dfrac{G'}{G} U^u U^y = 0 , \end{cases} \tag{12}$$

which can be easily integrated to give the trajectory

$$\begin{cases} u(\tau) = c^u \tau + u_0 , \\ v(\tau) = \dfrac{c_x}{2c^u} x(u(\tau)) + \dfrac{c_y}{2c^u} y(u(\tau)) , \\ x(\tau) = \dfrac{c_x}{c^u} \displaystyle\int_{u_0}^{u} \dfrac{\mathrm{d}u}{F^2} , \\ y(\tau) = \dfrac{c_y}{c^u} \displaystyle\int_{u_0}^{u} \dfrac{\mathrm{d}u}{G^2} , \end{cases} \tag{13}$$

where c^x, c^y and c^u are constants.

2·1. *Constants of the motion.* – In the general case the equations of motion (2) have a remarkable property of conservation related to the structure of the background under investigation.

If a Killing vector ξ^α of the background metric is given, then the following function:

$$C^{\xi} \overset{\text{def}}{=} P^{\alpha} \xi_{\alpha} + \frac{1}{2} \nabla_{\alpha} \xi_{\beta} S^{\alpha\beta} \tag{14}$$

is a constant of the motion (see [13] p. 327).

Therefore, in the case of the metric (5), the three coordinate directions of v, x, y, easily provide us with three constants of the motion: C^u, C^x, C^y, such that we can render explicit three of the components of **P**:

$$\left\{\begin{aligned} P^u &= C^u\,,\\ P^x &= \frac{1}{F^2}C^x - \frac{F'}{F}S^{ux}\,,\\ P^y &= \frac{1}{G^2}C^y - \frac{G'}{G}S^{uy}\,. \end{aligned}\right. \tag{15}$$

2$\cdot$2. *Background conditions.* – Now we have to impose a suitable set of supplementary conditions. From (8) we can calculate

$$R_{\alpha\beta\varrho\sigma}S^{\varrho\sigma} = 4FF''\delta_{x[\alpha}\delta_{\beta]u}S^{ux} + 4GG''\delta_{y[\alpha}\delta_{\beta]u}S^{uy} \tag{16}$$

and, consequently, it follows from $(2)_1$ that, under the hypothesis $S^{ux} = S^{uy} = 0$, it results $\mathrm{D}P^\alpha/\mathrm{d}\tau_U = 0$. Therefore we are led to choose the following background condition:

$$S^{u\alpha} = 0\,. \tag{17}$$

This is indeed a condition of the kind (3), with **V** a null vector, suggested by the structure of the background.

Now $(2)_1$ and (16) give

$$\frac{\mathrm{D}}{\mathrm{d}\tau_U}P^\alpha = 0\,. \tag{18}$$

For $\alpha = u$, eq. (18) gives us back the constant of the motion $(15)_1$: $P^u = C^u$. Therefore, under condition (17), from the equation of motion $(2)_2$ we have

$$P^\alpha U^u - C^u U^\alpha = 0\,. \tag{19}$$

Multiplicating by U_α, (19) gives $U^u = C^u/m$, and, consequently,

$$P^\alpha = mU^\alpha. \tag{20}$$

Thus, from $(2)_2$ it follows:

$$\frac{\mathrm{D}}{\mathrm{d}\tau_U}S^{\alpha\beta} = 0\,. \tag{21}$$

Moreover, from (18) and (20) it also results

$$\frac{\mathrm{D}}{\mathrm{d}\tau_U}m = 0\,, \qquad \frac{\mathrm{D}}{\mathrm{d}\tau_U}U^\alpha = 0\,. \tag{22}$$

The resulting scheme of the spinning particle in the field of a gravitational plane wave is the following.

The momentum vector and the spin tensor are parallel-transported along the center line (from eqs. (18), (21)).

There is absence of inner motion (eq. (20)).

The mass is conserved along the center line (eq. $(22)_1$).

The world line is a geodesic (eq. $(22)_2$).

2.3. *Traversing the wavefront.* – The metric (5) matches with the flat metric on the hypersurface Σ: $t = z$, where the Riemann tensor has a jump discontinuity: $[R_{\alpha\beta\varrho\sigma}] \neq 0$, under the hypothesis that the metric is globally of class (C^2, piecewise C^3). In particular, $F = G = 1$ on Σ, and $[F] = [G] = [F'] = [G'] = 0$, $[F''] = -[G'']$.

Σ is called the wavefront of the gravitational wave, and it is a null hypersurface (for a general treatment of gravitational wavefronts, see, for example: [7, 16]). Let l_α denote the normal vector to the wavefront; in our coordinates: $l_u = \sqrt{2}$, $l_v = l_x = l_y = 0$.

The jump of the curvature tensor is, from (8),

$$[R_{\alpha\beta\varrho\sigma}] = 4[F''](g_{x[\alpha}g_{\beta]v}g_{v[\varrho}g_{\sigma]x} - g_{y[\alpha}g_{\beta]v}g_{v[\varrho}g_{\sigma]y}) \tag{23}$$

or, in the 6-dimensional notation of (9), if $\sigma \overset{\text{def}}{=} [F'']$,

$$[R_{AB}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sigma & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma \end{pmatrix}. \tag{24}$$

Let $U^\alpha l_\alpha = -1$, let $\mathbf{U}$ and $S^{\alpha\beta}$ be continuous across the wavefront, and let δP_α denote the discontinuity of the first derivatives of the momentum vector, such that the compatibility conditions (see, for example, [17]) hold:

$$[\partial_\alpha P_\beta] = l_\alpha \delta P_\beta. \tag{25}$$

Then, from $(2)_1$, the following deviation formula results:

$$\delta P_\alpha = [F'']\{(\delta_{\alpha x} U^u - \delta_{\alpha u} U^x) S^{ux} - (\delta_{\alpha y} U^u - \delta_{\alpha u} U^y) S^{uy}\}. \tag{26}$$

Therefore, traversing the wavefront would generally cause a discontinuity to appear in the motion, at the first derivatives of the momentum vector, given by eq. (26).

One can easily verify that $l^\alpha \delta P_\alpha = U^\alpha \delta P_\alpha = 0$, *i.e.* the effect of traversing the wavefront is "transverse".

If (17) holds, it results $[\mathrm{D}P_\alpha / \mathrm{d}\tau_U] = 0$ and $\delta P_\alpha = 0$. Therefore, such discontinuity does not arise under our background condition.

REFERENCES

[1] MATHISSON M., *Acta Phys. Polon.*, **6** (1937) 167.
[2] TAUB A. H., *J. Math. Phys.*, **5** (1964) 112.
[3] PAPAPETROU A., *Proc. R. Soc. London, Ser. A*, **209** (1951) 248.
[4] TULCZYJEW W., *Acta Phys. Polon.*, **18** (1959) 393.
[5] DIXON W. G., *Nuovo Cimento*, **34** (1964) 318.
[6] FERRARESE G., *Lezioni di relatività generale* (Pitagora ed., Bologna) 1994.
[7] PIRANI F. A., *Phys. Rev.*, **105** (1957) 1089.
[8] RAGUSA S. and BAILYN M., *Gen. Relativ. Gravit.*, **27** (1995) 163.

[9] DIRAC P. A. M., *Proc. R. Soc. London, Ser. A*, **167** (1938) 148.
[10] EHLERS J., and RUDOLPH E., *Gen. Relativ. Gravit.*, **8** (1977) 197.
[11] FERRARESE G., BINI D., STAZI L., GEMELLI G. and RICCI M., *Nuovo Cimento B*, **118** (1996) 217.
[12] CORINALDESI E. and PAPAPETROU A., *Proc. R. Soc. London, Ser. A*, **209** (1951) 259.
[13] CARMELI M., *Classical Fields: General Relativity and Gauge Theory* (J. Wiley & Sons, New York-Chichester-Brisbane-Toronto-Singapore) 1982.
[14] TOD K. P., DE FELICE F. and CALVANI M., *Nuovo Cimento B*, **34** (1976) 365.
[15] MISNER C. W., THORNE K. S. and WHEELER J. A., *Gravitation* (W. H. Freeman and Company, New York, N.Y.) 1973.
[16] LICHNEROWICZ A., *Ann. Matem.*, **50** (1960) 2.
[17] CATTANEO C., *Ist. Lomb. Accad. Sci. Lett. Rend. A*, **112** (1978) 139.