Structural transitions in ion crystals: a platform to study criticality

Low dimensional structures

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Observation of Ordered Structures of Laser-Cooled Ions in a Quadrupole Storage Ring

I. Waki,[a] S. Kassner, G. Birkl, and H. Walther

Max-Planck-Institut für Quantenoptik, D-8046 Garching bei München, Federal Republic of Germany

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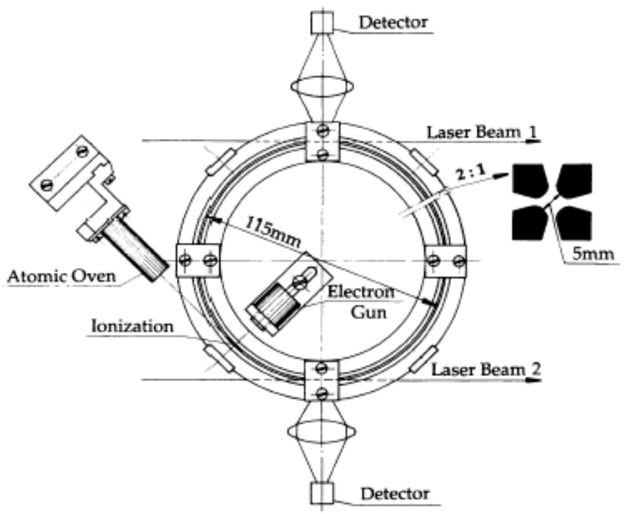

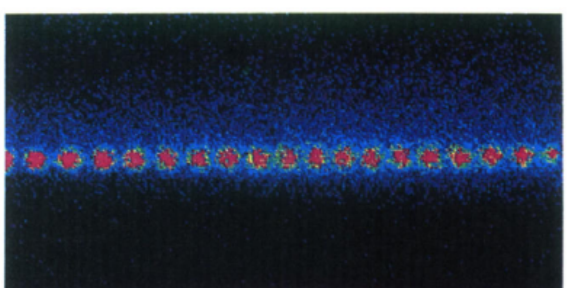

FIG. 3. Spatially resolved image of the fluorescence light emitted by an ordered structure of 19 ions forming a linear string. The distance between the ions is 33 ± 1 μm. The image

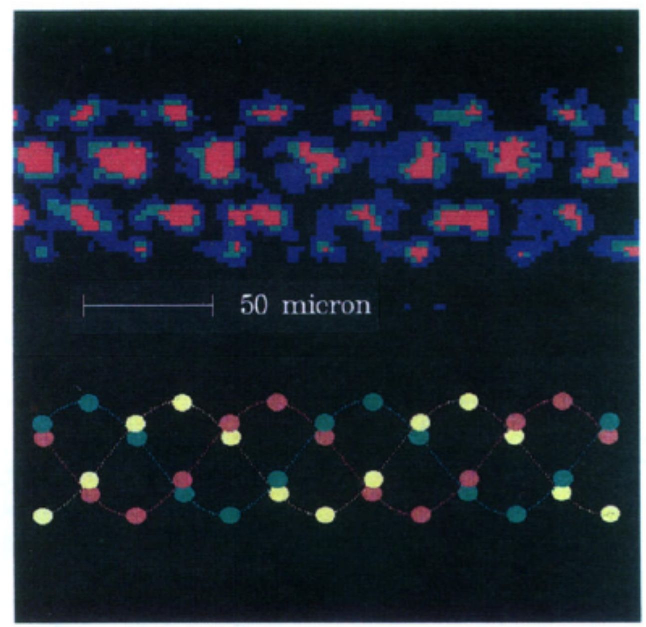

FIG. 6. Helical structure of $^{24}Mg^+$ ions with a diameter of 63 ± 2 μm. The experimental image (top) corresponds to three interwoven helices (shown in different colors, bottom). The closely appearing pairs of ions are sitting on opposite sites, re-

Phase diagram

Multiple-shell structures of laser-cooled $^{24}Mg^{+}$ ions in a quadrupole storage ring

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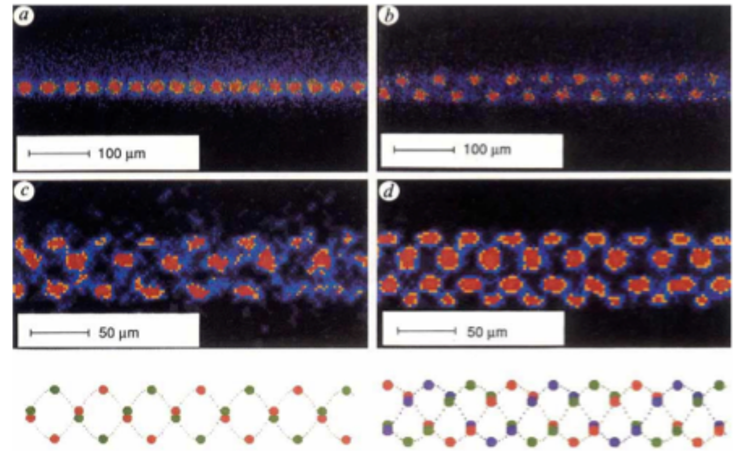
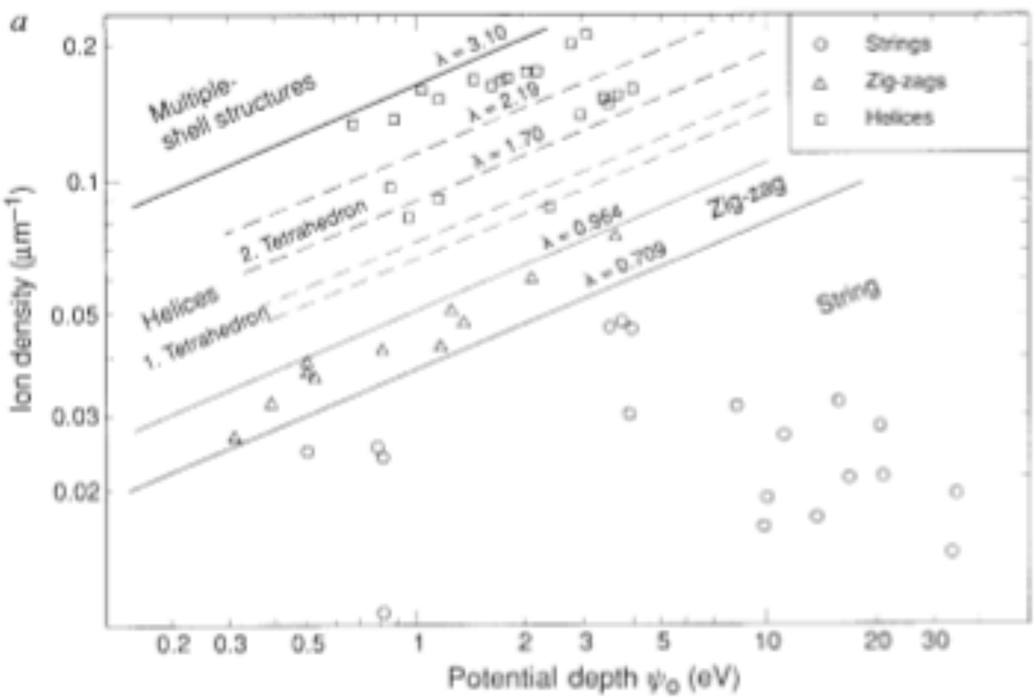
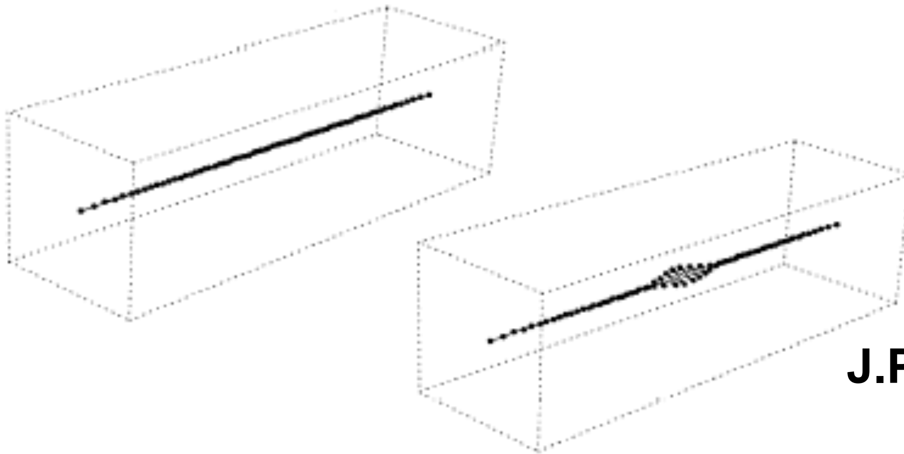
Max-Planck-Institut für Quantenoptik, Garching bei München, Germany

Nature 357, 310 (1992)





Structural instability: Linear-Zigzag

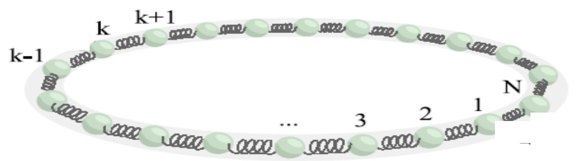


J.P. Schiffer, PRL 1993.

decrease transverse confinement ν_t

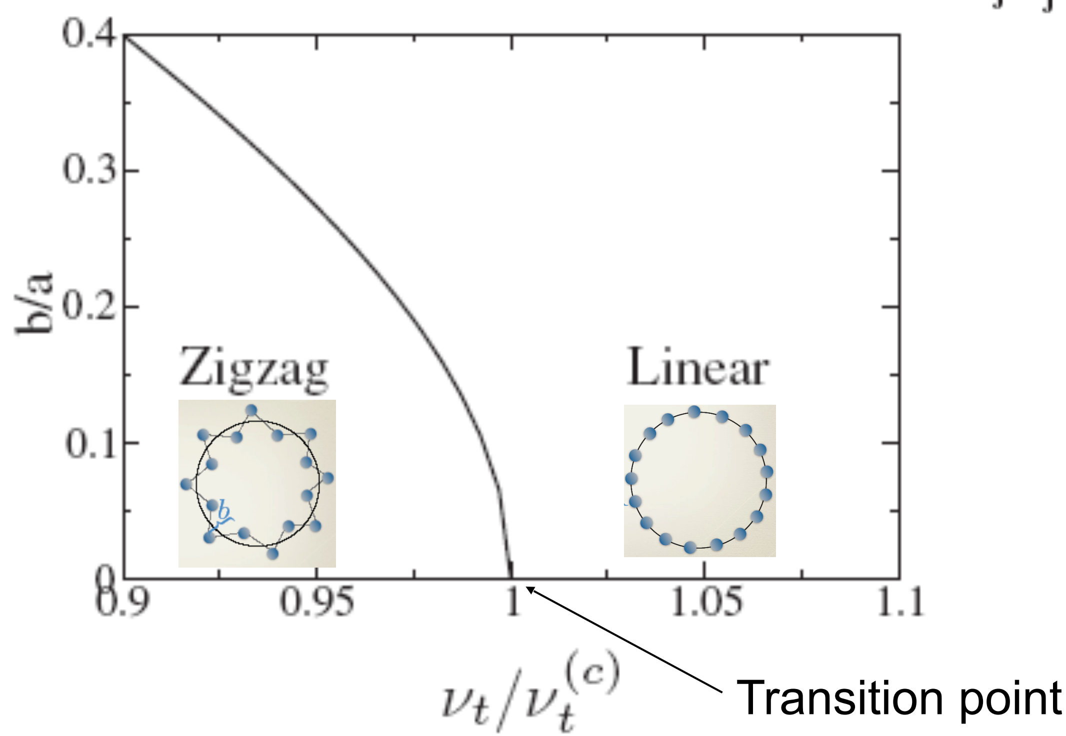
J.Eschner and coworkers, Barcelona, 2007.

Instability for ions in a ring



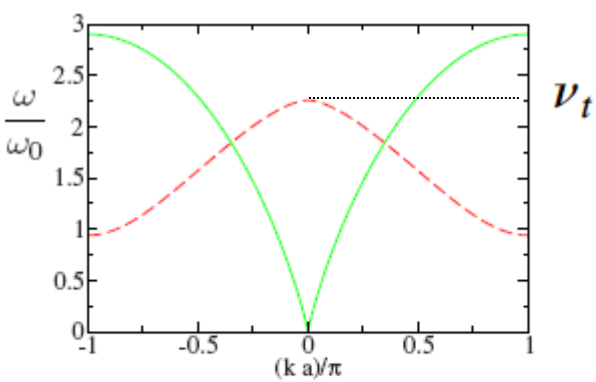
Fixed interparticle distance a
(thermodynamic limit)

Find transverse displacement b with $\left.\frac{\partial V}{\partial \mathbf{r_j}}\right|_{\mathbf{r_j}=\mathbf{r_j}^{(0)}} = 0$



Normal modes – Ring

No axial confinement: periodic distribution
Modes are phononic waves with quasimomentum k in BZ

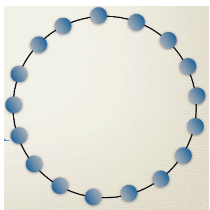


Spectra of excitations

transverse motion

$$\omega_\perp(k)^2 = \nu_t^2 - 2\left(\frac{2Q^2}{ma^3}\right)\sum_{j=1}^{N}\frac{1}{j^3}\sin^2\frac{jka}{2}$$

Normal modes – linear chain



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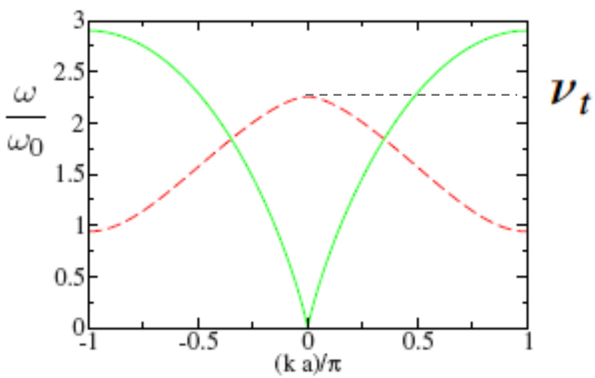

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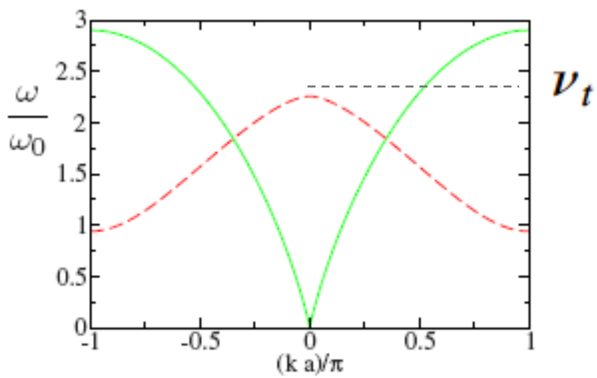
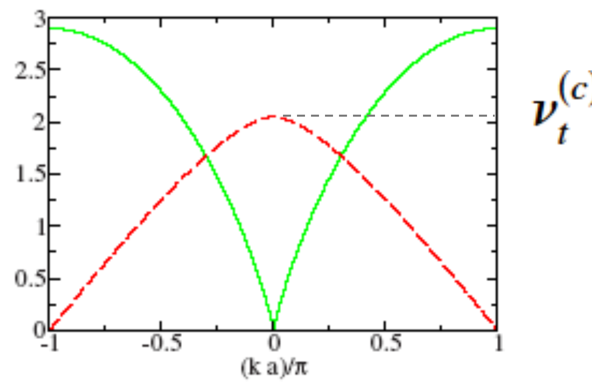




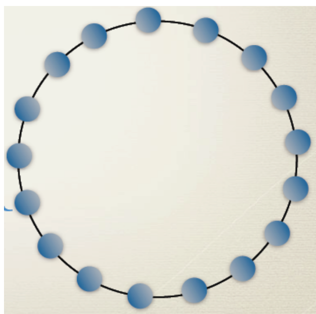
Critical value:

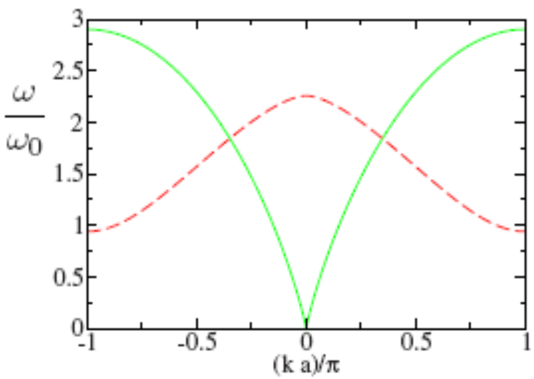
$$\nu_t^{(c)\ 2} = 2\left(\frac{2Q^2}{ma^3}\right)\sum_{j=1}^{N}\frac{1}{j^3}\sin^2\frac{j\ \pi}{2} \rightarrow \frac{Q^2}{ma^3}\frac{7}{2}\zeta(3)$$

Transition linear-zigzag

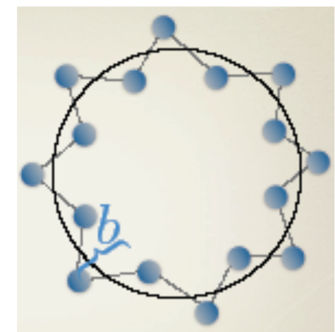
decrease transverse confinement ν_t

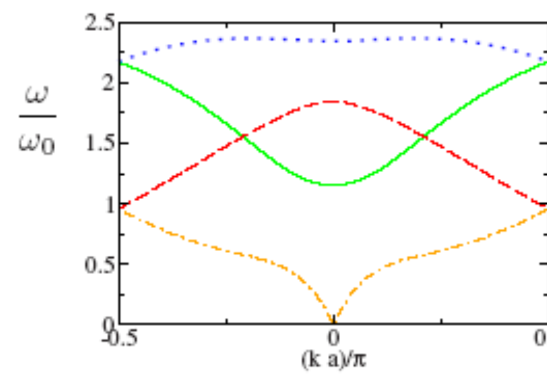
Linear chain





Zigzag





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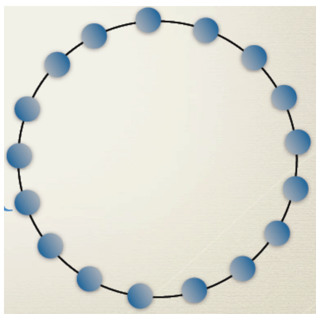
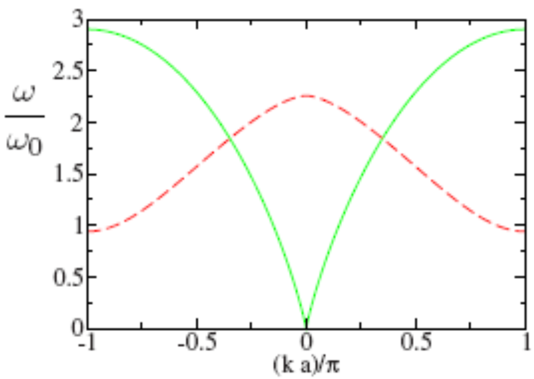
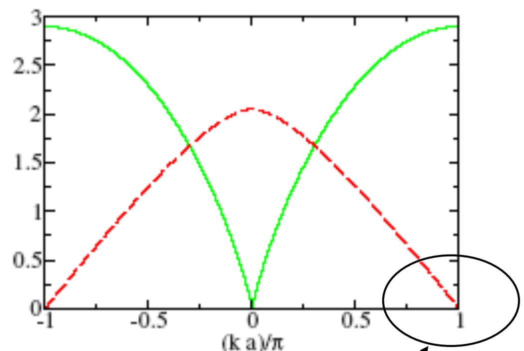
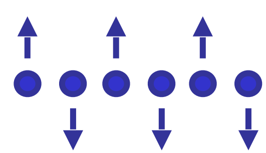




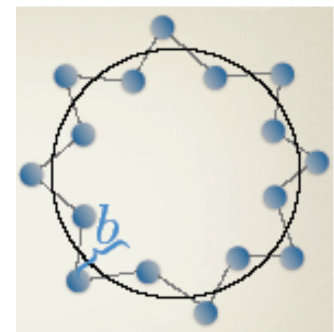
Transition point

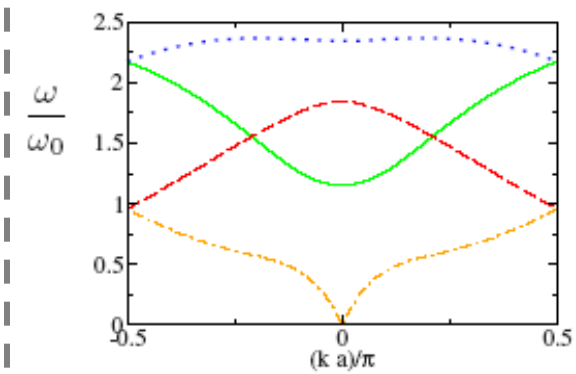


Zigzag mode

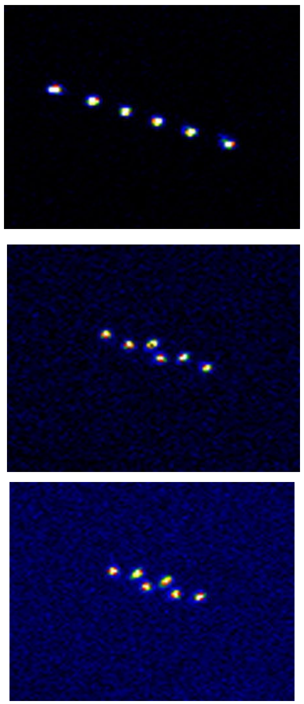


Zigzag





Linear-Zigzag: second-order phase transition?



Educated guess:

Symmetry breaking: line to plane

Order parameter: Equilibrium distance from the axis

Control field: Transverse frequency

Soft mode: Zigzag mode

Mapping to Landau's model-I

Simplified version: Longitudinal positions fixed only transverse oscillations in one direction.

$$\omega_\perp^2(k) = \nu_t^2 - 4\omega_0^2 \sum_{j=1}^{N} \frac{1}{j^3} \sin^2 \frac{jka}{2}$$

$$\hat{\psi}_k = \sum_j \psi_j \mathrm{e}^{ijka}/\sqrt{N}$$

zigzag mode: wavelength $\lambda_0 \equiv 2\pi/k_0 = 2a$

close to the instability $V' \simeq \frac{1}{2} \sum_{k>0} r_k \hat{\psi}_k \hat{\psi}_{-k} + V^{(4)}$

$$r_k = m\omega_\perp^2(k) a^2$$

Mapping to Landau's model-II

close to the instability $V' \simeq \frac{1}{2}\sum_{k>0} r_k \hat{\psi}_k \hat{\psi}_{-k} + V^{(4)}$

$\nu_t > \nu_t^{(c)}$ all coefficients $r_k = m\omega_\perp^2(k)a^2$ are positive:

the mean value of all $\hat{\psi}_k$ vanishes —> Linear chain

$\nu_t < \nu_t^{(c)}$ the effective potential to minimize is

$$V_0' = \frac{1}{2} r_{k_0} |\hat{\psi}_{k_0}|^2 + A_4 |\hat{\psi}_{k_0}|^4$$

$$\hat{\psi}_{k_0} \propto \pm\sqrt{\nu_t^{(c)} - \nu_t}$$

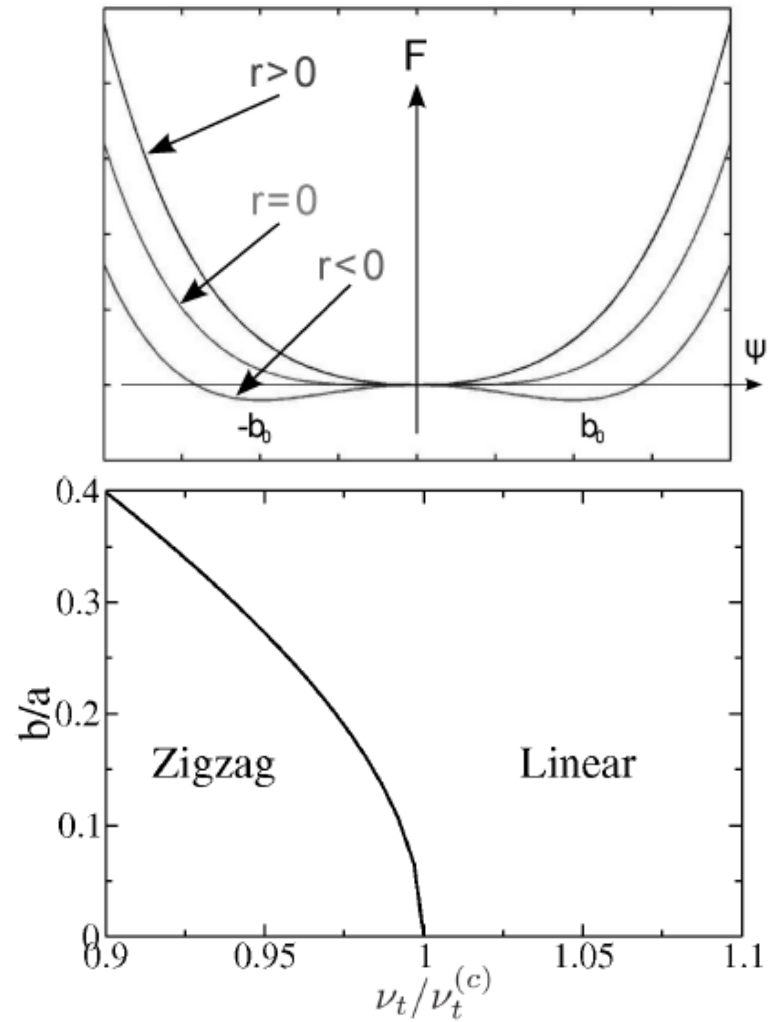
Mapping to Landau's model-III

$$V_0' = \frac{1}{2} r_{k_0} |\hat{\psi}_{k_0}|^2 + A_4 |\hat{\psi}_{k_0}|^4$$

$$\hat{\psi}_{k_0} \propto \pm\sqrt{\nu_t^{(c)} - \nu_t}$$

Transverse equilibrium positions

$$b_j = \pm(-1)^j C \sqrt{\nu_t^{(c)} - \nu_t}$$



Why taking only the zigzag mode and ignore the other transverse modes?

Consider

$$F = \frac{1}{2} r_1 \psi_1^2 + \frac{1}{2} r_2 \psi_2^2 + A_{11} \psi_1^4 + A_{22} \psi_2^4 + A_{12} \psi_1^2 \psi_2^2$$

with $r_1 < r_2$ and $A_{ij} > 0$

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with $r_1 < r_2$ and $A_{ij} > 0$

For $0 < r_1 < r_2$ then $\psi_1 = \psi_2 = 0$

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For $0 < r_1 < r_2$ then $\psi_1 = \psi_2 = 0$

For $r_1 < 0 < r_2$ then $\psi_2 = 0$ and $\psi_1 = \pm\sqrt{-r_1/4A_{11}}$

the quadratic term for mode 2 is

$r_2 + 2A_{12}\psi_1^2$ >0

Why taking only the zigzag mode and ignore the other transverse modes?

Consider

$$F = \frac{1}{2}r_1\psi_1^2 + \frac{1}{2}r_2\psi_2^2 + A_{11}\psi_1^4 + A_{22}\psi_2^4 + A_{12}\psi_1^2\psi_2^2$$

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For $0 < r_1 < r_2$ then $\psi_1 = \psi_2 = 0$

For $r_1 < 0 < r_2$ then $\psi_2 = 0$ and $\psi_1 = \pm\sqrt{-r_1/4A_{11}}$

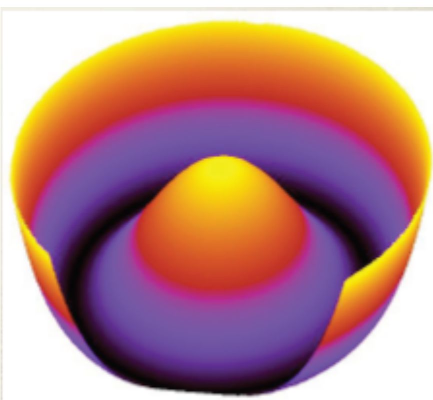
For $r_2 < 0$ (small) the quadratic term is still

$r_2 + 2A_{12}\psi_1^2$ >0 thus $\psi_2 = 0$

Cylindrical potential

Ansatz: the zigzag mode is the soft mode.
One finds the effective potential for the zigzag mode:

$$V^{\mathrm{soft}} = \mathcal{V}[(\Psi_0^y)^2 + (\Psi_0^z)^2] + A[(\Psi_0^y)^2 + (\Psi_0^z)^2]^2$$

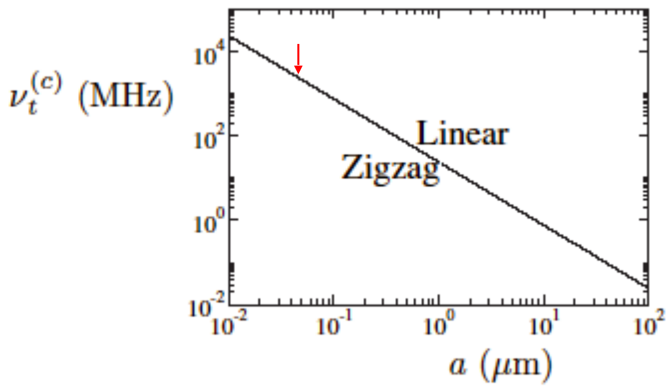


$$\mathcal{V} = \frac{m}{2}\beta_0 = \frac{1}{2}m(\nu_t^2 - \nu_t^{(c)2})$$

$$A = \frac{3}{2}\frac{31}{32}\zeta(5)\frac{Q^2}{a^5}$$

The other modes are stably trapped by a harmonic potential

Scaling laws at the structural phase transition



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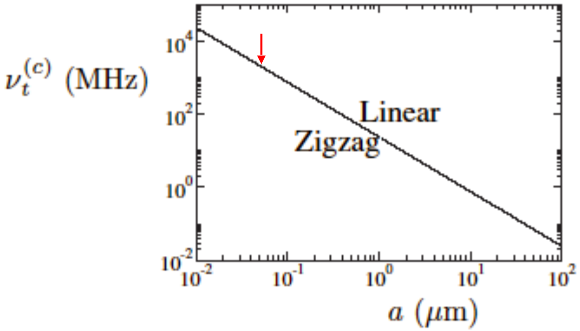
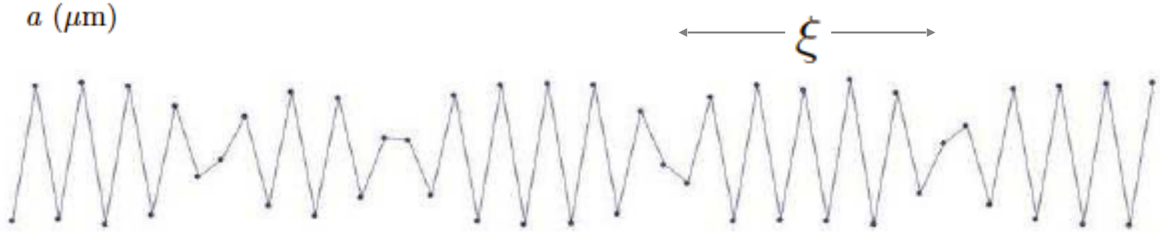


Correlation length

$$\xi \sim (\nu_t - \nu_t^{(c)})^{-1/2}$$



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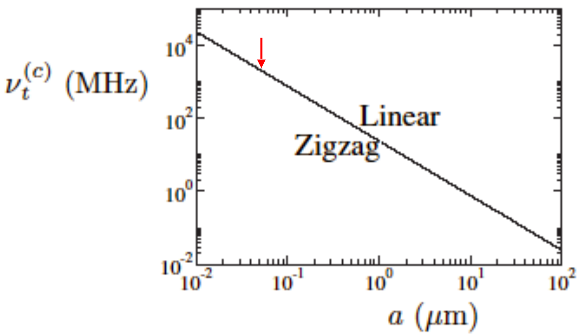

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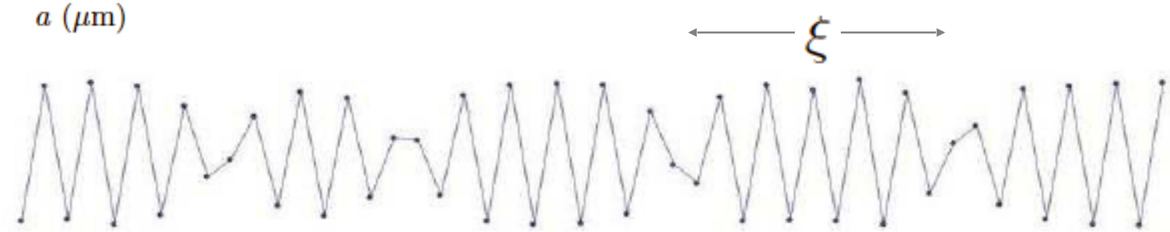
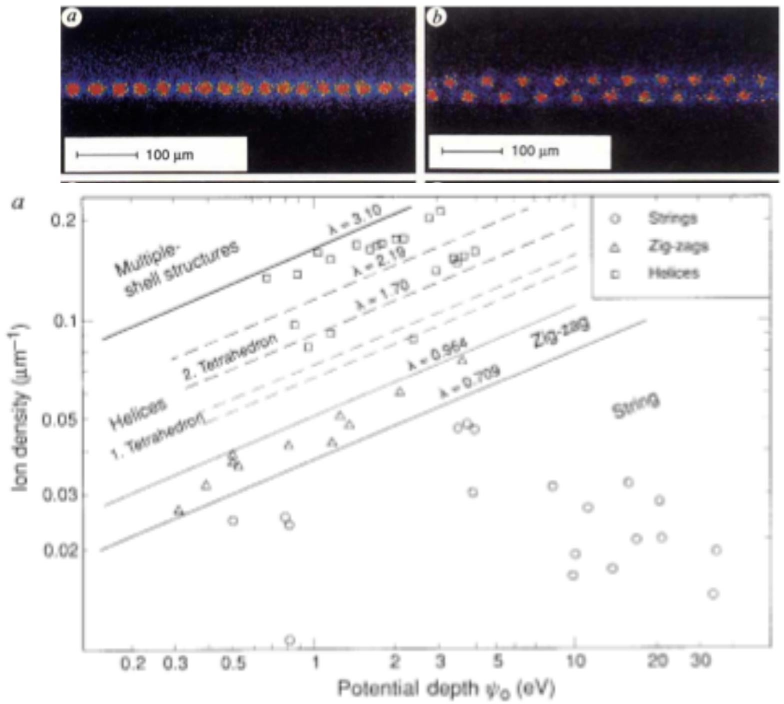


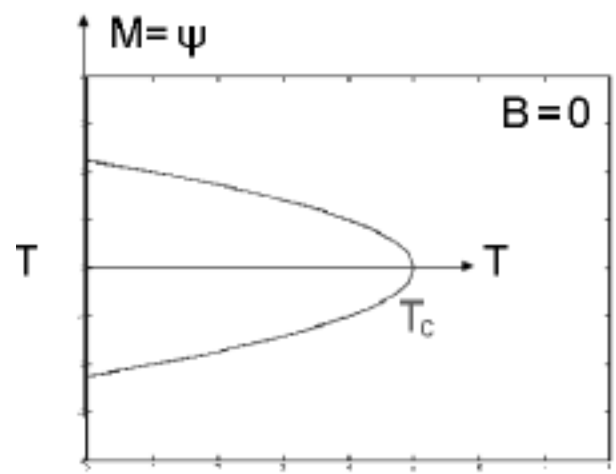
Relaxation time

$$\tau \sim (\nu_t - \nu_t^{(c)})^{-1}$$

Linear-Zigzag like Ferromagnetism



Same critical behaviour as
a ferromagnet at B=0



Fluctuations

In 1 D: no long-range order for nonzero temperature

But if the chain is shorter than the correlation length it will look ordered in one of the zigzag phases

$\xi \sim (\nu_t - \nu_t^{(c)})^{-1/2}$ > length *L*

Fluctuations

In 1 D: no long-range order for nonzero temperature

But if the chain is shorter than the correlation length
it will look ordered in one of the zigzag phases

$\xi \sim (\nu_t - \nu_t^{(c)})^{-1/2}$ > length *L*

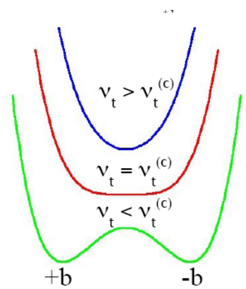
T=0 and $"\hbar" = 0$ $\longrightarrow$ no fluctuations
transition at $r = 0$
$\nu_t = \nu_t^{(c)}$

Quantum fluctuations

decrease transverse confinement ν_t

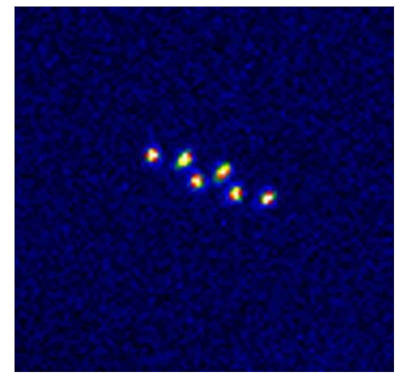
linear

disordered phase



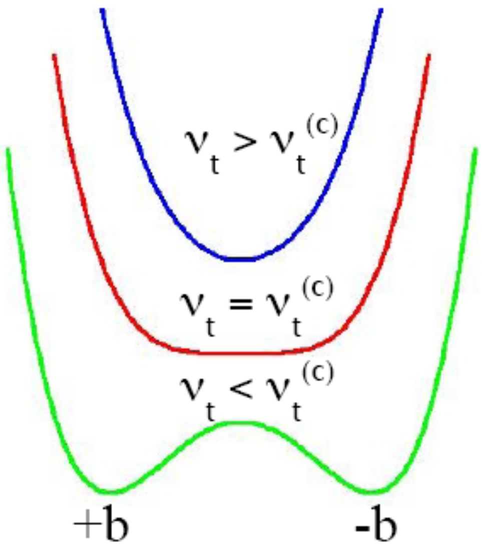
tunneling between the minima

zigzag



Mapping to an Ising model with transverse field.

Quantum fluctuations at the mechanical instability



Soft mode: $y_j^{\mathrm{soft}} = (-1)^j y_0$

Modes close to the instability: $y_j = (-1)^j a\phi_j$

Partition function $Z = \int \mathcal{D}\phi\, e^{-S[\phi]/\hbar}$

Euclidean action

$$S[\phi] = \int_0^{\hbar\beta} d\tau \sum_{j=1}^{N} \left[\frac{1}{2}ma^2(\partial_\tau\phi_j)^2 + V_0(\phi_j) + \frac{1}{2}K(\phi_j - \phi_{j+1})^2\right]$$

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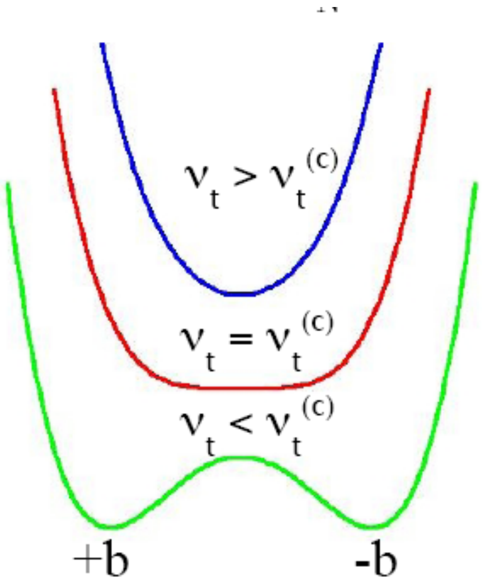

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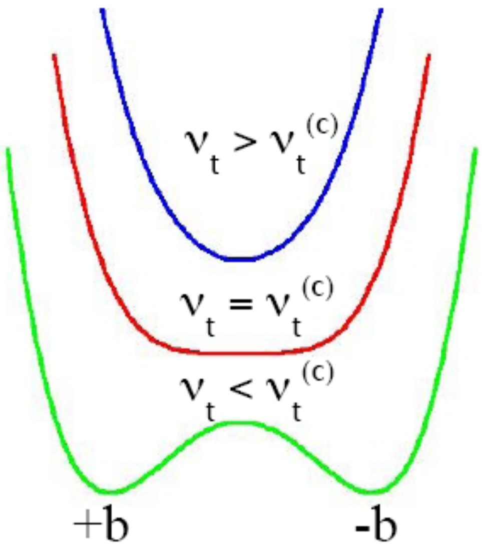
Fluctuations (1+1 dimensions)

Quantum fluctuations at the mechanical instability



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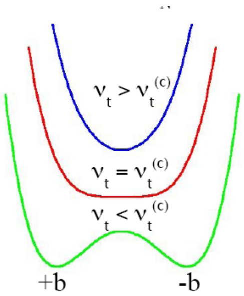
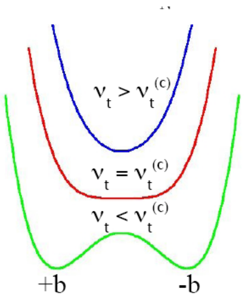
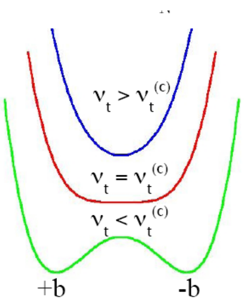
local potential

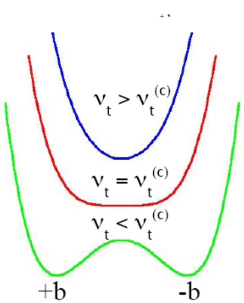
$$V_0(\phi) = -\frac{1}{2} m (\nu_c^2 - \nu_t^2) a^2 \phi^2 + \frac{1}{4} g a^4 \phi^4$$

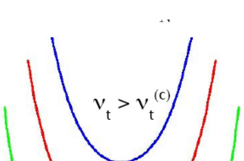
Mapping to an Ising model with transverse field











Two wells: two values of the spin

$$\phi_j = \phi_0 \sigma_j^z + \delta\phi_j$$

$$Z \approx Z_0 \int \mathcal{D}\sigma \exp\left(-S_I[\sigma]/\hbar\right)$$

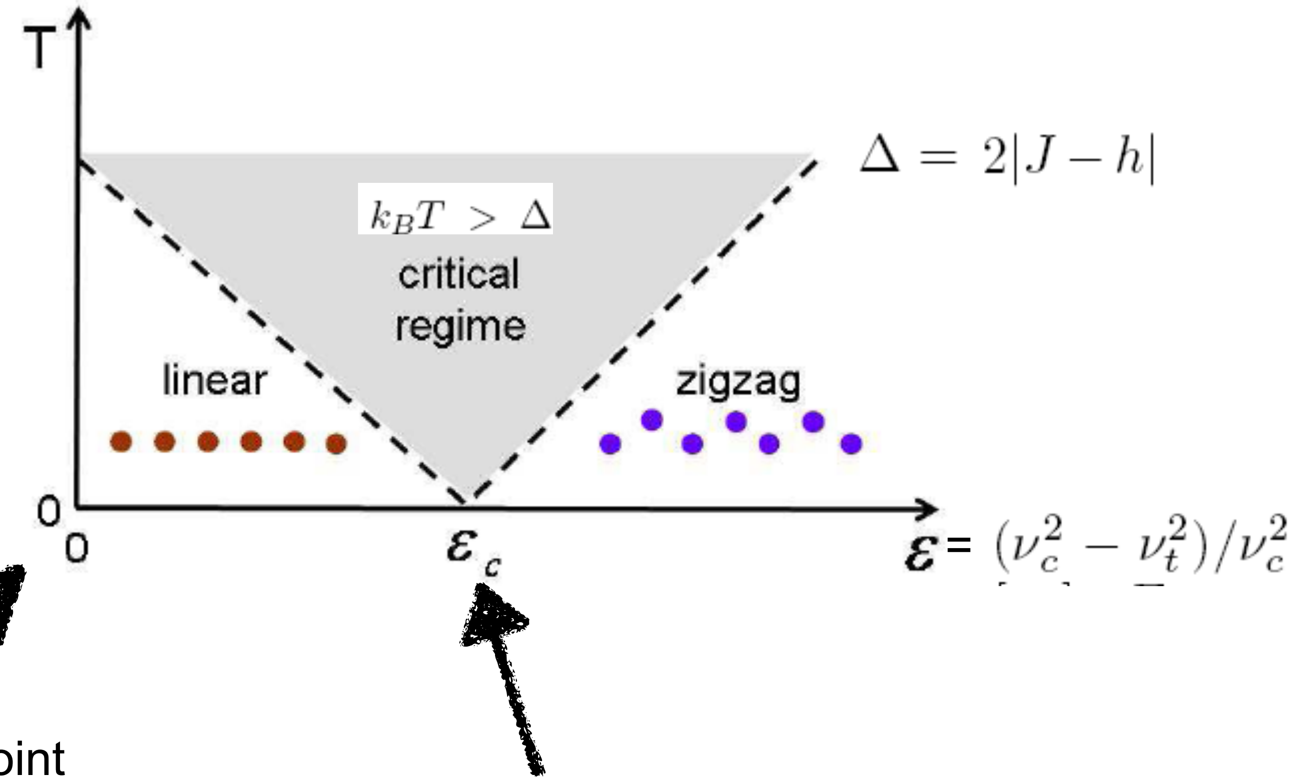
Effective Hamiltonian

$$H_I = -\sum_{j=1}^{N} \left(J\sigma_j^z \sigma_{j+1}^z + h\sigma_j^x\right)$$

$h \approx C_h \left(U_P U_K^2\right)^{1/3}$ transverse field (tunneling)

$J = K\phi_0^2 = C_J U_P \varepsilon$ exchange coupling (Coulomb interaction)

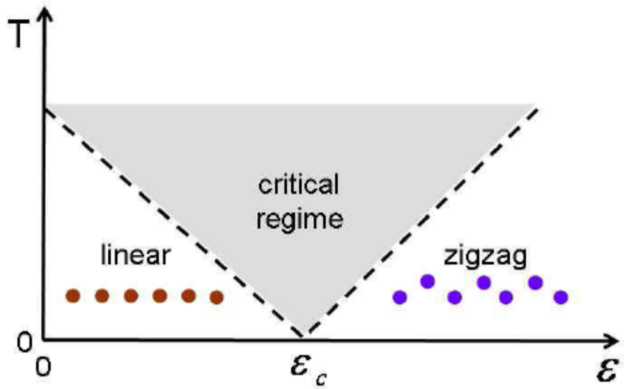
Phase diagram



classical stability point

quantum critical point at J=h

Experimental parameters



Resolution for trap frequency

$$\delta\nu \approx 10^{-4} \frac{1}{2(n_A a_0)^{2/3}} \nu_c$$

Temperatures required:

$$T[\mathrm{mK}] \ll 0.25 \left(\frac{1}{n_A^2 a_0^5} \right)^{1/3}$$

a_0 interparticle distance in units of μm

n_A atomic number

DMRG results

$$\mathcal{H} = \frac{1}{2}\sum_{j=1}^{L}\left[\pi_j^2 - \varepsilon\phi_j^2 + (\phi_j - \phi_{j+1})^2 + 2g\phi_j^4\right]$$

$$[\phi_j, \pi_\ell] = i\tilde{\hbar}\delta_{j,\ell}$$

$$\tilde{\hbar} \sim \sqrt{\frac{U_K}{U_P}}$$

Table 1 Resume of the computed critical exponents.

	Quantity	Computed	Theory [7]
η	Anomalous dimension	0.258 ± 0.012	0.25
β	Spont. magnetization	0.126 ± 0.011	0.125
ν	Correlation length	1.03 ± 0.05	1
c	Central charge	0.487 ± 0.015	0.5

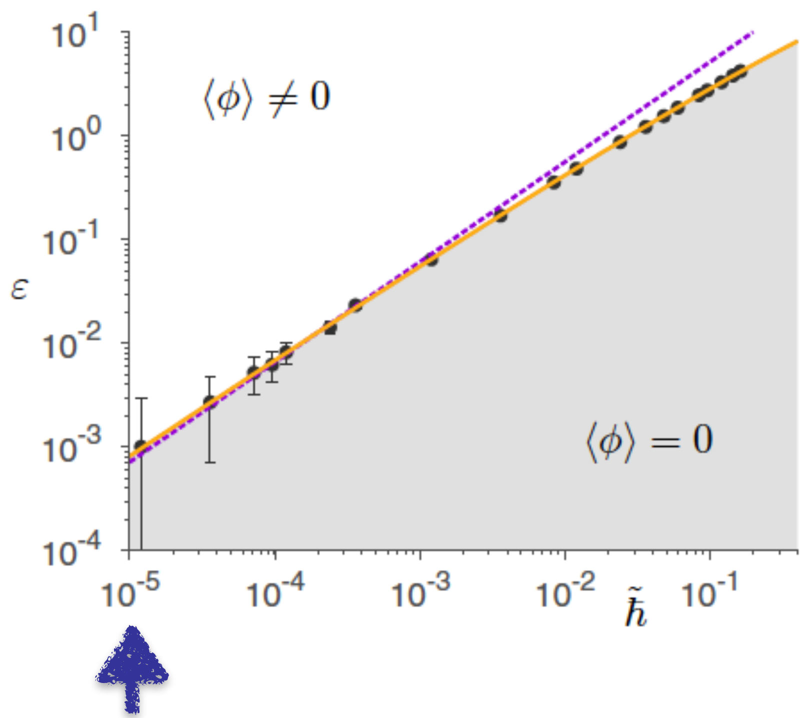
DMRG results

$$\mathcal{H} = \frac{1}{2}\sum_{j=1}^{L}\left[\pi_j^2 - \varepsilon\phi_j^2 + (\phi_j - \phi_{j+1})^2 + 2g\phi_j^4\right]$$

$$[\phi_j, \pi_\ell] = i\tilde{\hbar}\delta_{j,\ell}$$

shift of the critical frequency from the mean-field value

$$\tilde{\hbar} \sim \sqrt{\frac{U_K}{U_P}}$$



Ions

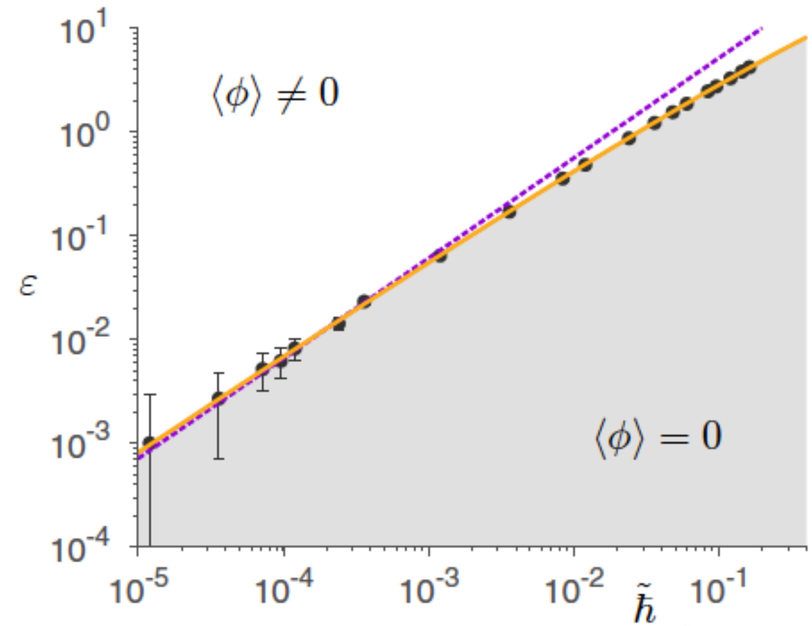
DMRG results

$$\mathcal{H} = \frac{1}{2}\sum_{j=1}^{L}\left[\pi_j^2 - \varepsilon\phi_j^2 + (\phi_j - \phi_{j+1})^2 + 2g\phi_j^4\right]$$

$$[\phi_j, \pi_\ell] = i\tilde{\hbar}\delta_{j,\ell}$$

shift of the critical frequency from the mean-field value

$$\tilde{\hbar} \sim \sqrt{\frac{U_K}{U_P}}$$



Dipolar gases

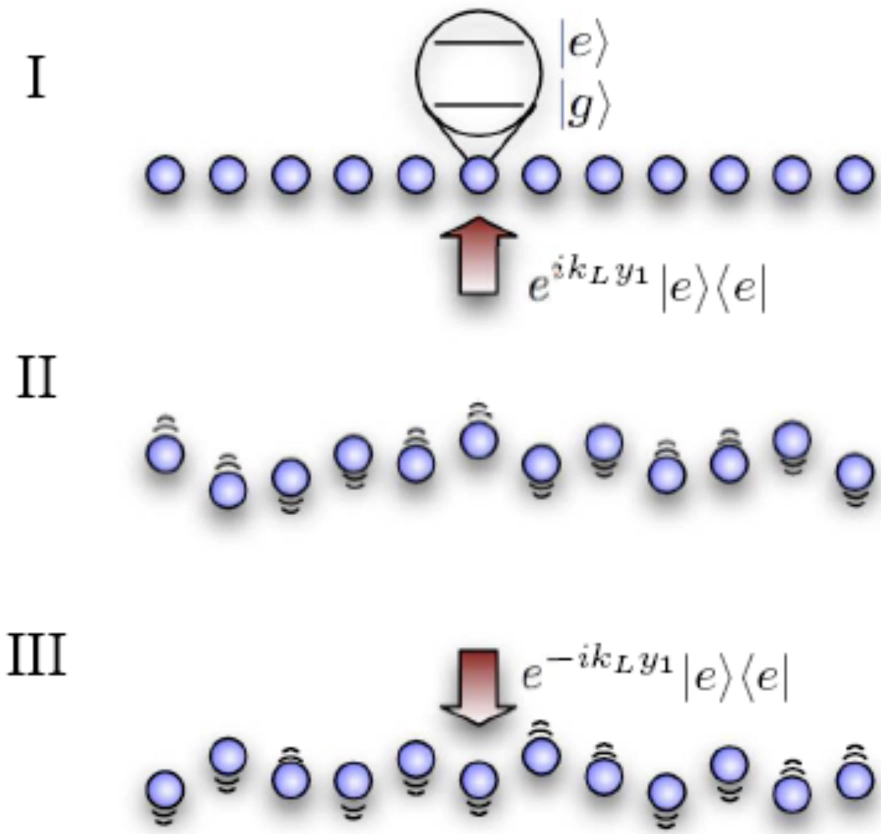
Measuring the critical behaviour

1) Ramsey interferometry

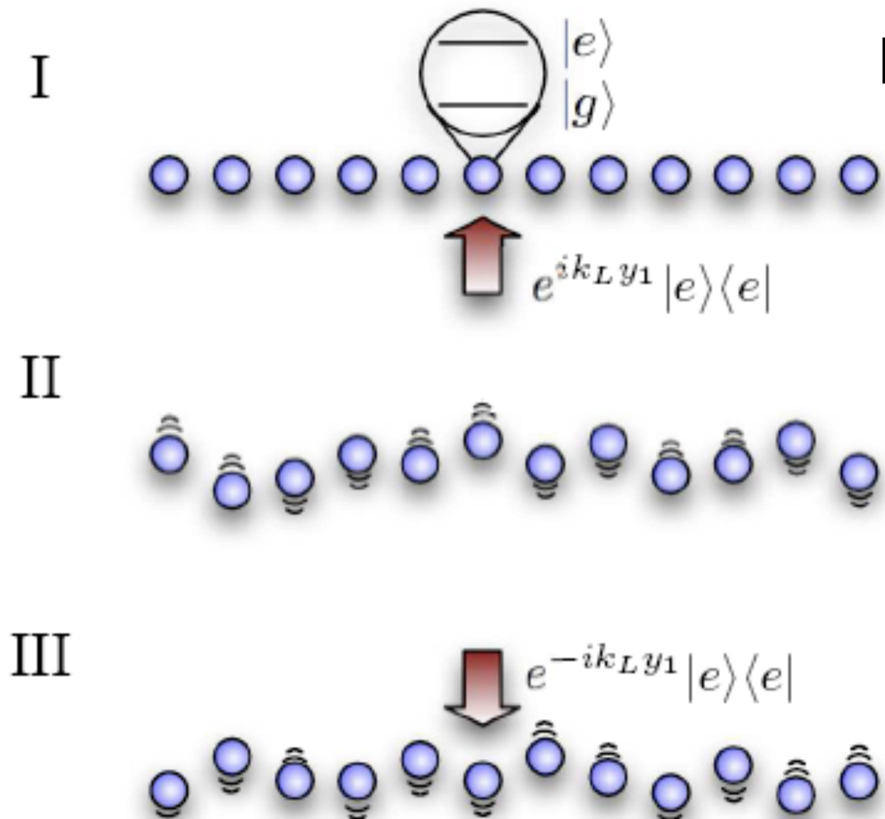
2) Quenches

3) …

Ramsey interferometry

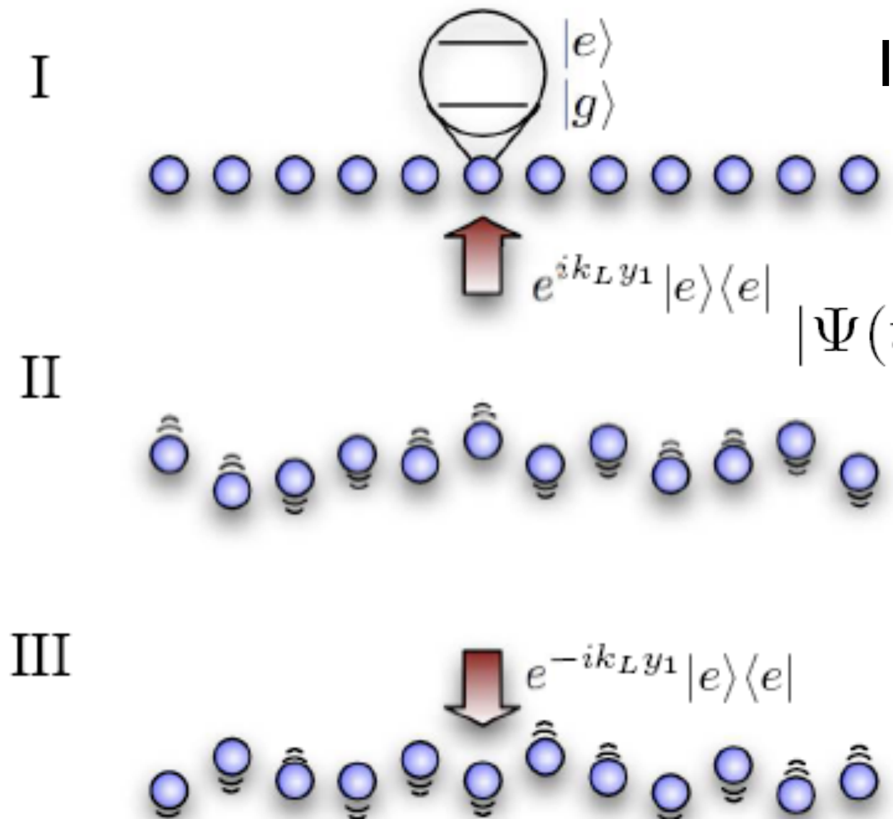


Ramsey interferometry



Initial state: $|\Psi(0)\rangle = |g, 0\rangle$

Ramsey interferometry



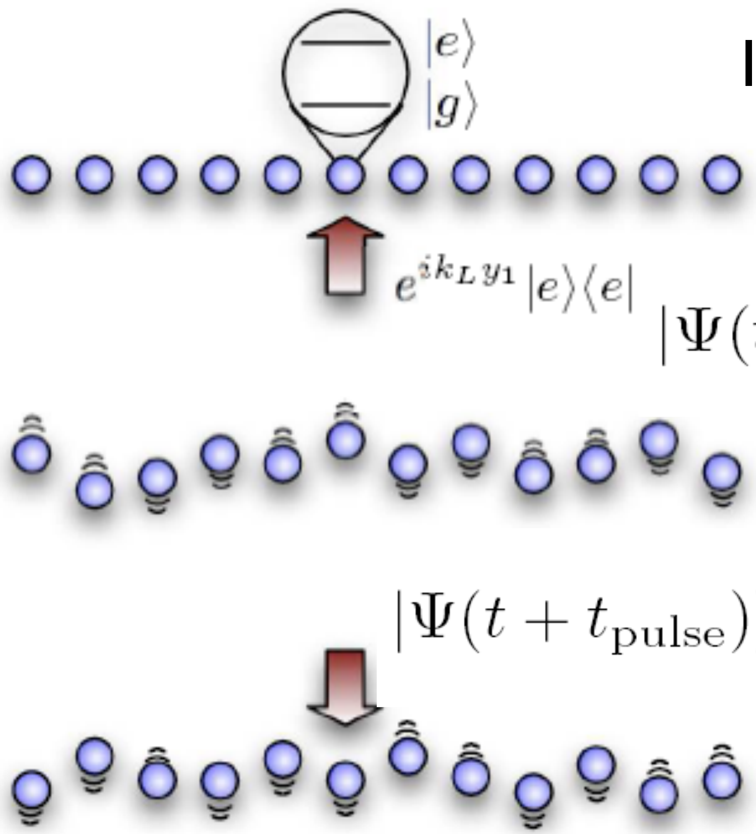
Initial state: $|\Psi(0)\rangle = |g, 0\rangle$

$$|\Psi(t_{\text{pulse}})\rangle = \frac{1}{\sqrt{2}}\left(|g, 0\rangle + |e, \alpha\rangle\right)$$

Ramsey interferometry

I

Initial state: $|\Psi(0)\rangle = |g, 0\rangle$



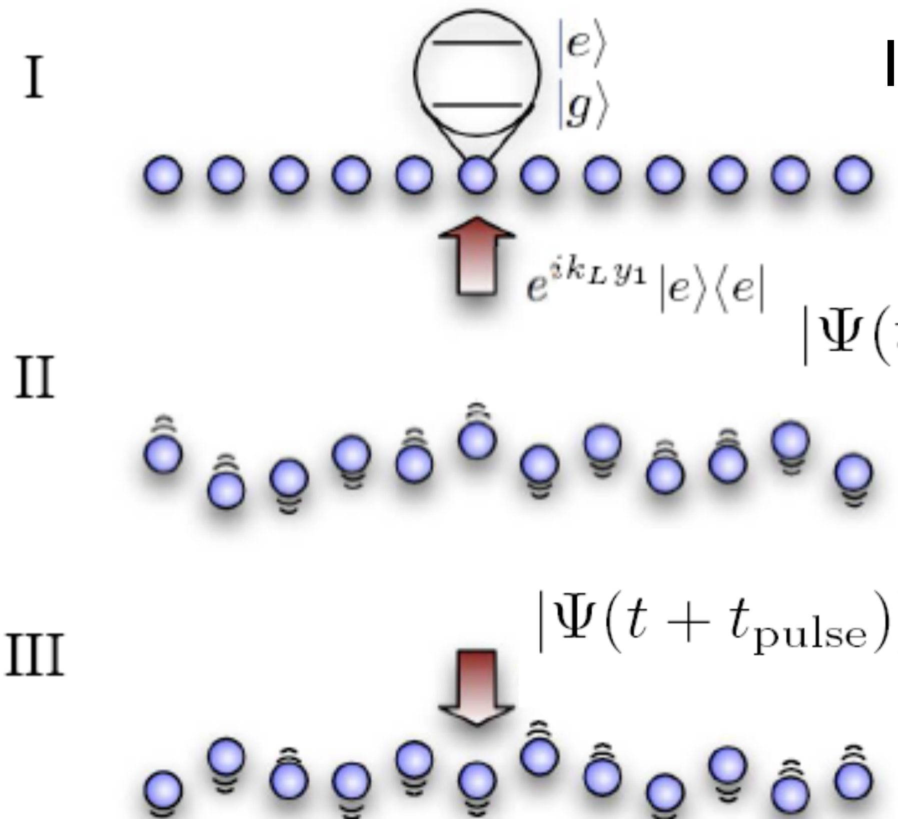
II

$$|\Psi(t_{\mathrm{pulse}})\rangle = \frac{1}{\sqrt{2}}\left(|g, 0\rangle + |e, \alpha\rangle\right)$$

III

$$|\Psi(t + t_{\mathrm{pulse}})\rangle = \frac{1}{\sqrt{2}}\left(|g, 0\rangle + \mathrm{e}^{\mathrm{i}\phi}|e, \alpha \mathrm{e}^{-\mathrm{i}\nu t}\rangle\right)$$

Ramsey interferometry



Initial state: $|\Psi(0)\rangle = |g,0\rangle$

$$|\Psi(t_{\text{pulse}})\rangle = \frac{1}{\sqrt{2}}\left(|g,0\rangle + |e,\alpha\rangle\right)$$

$$|\Psi(t+t_{\text{pulse}})\rangle = \frac{1}{\sqrt{2}}\left(|g,0\rangle + \mathrm{e}^{\mathrm{i}\phi}|e,\alpha\mathrm{e}^{-\mathrm{i}\nu t}\rangle\right)$$

Probability to be in the ground state:

$$\mathcal{P}_g(t) = \frac{1}{2}\left[1 + \mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}\phi}\mathcal{S}(t)\right\}\right]$$

Interferometric signal

Probability to be in the ground state:

$$\mathcal{P}_g(t) = \frac{1}{2}\left[1 + \mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}\phi}\mathcal{S}(t)\right\}\right]$$

Visibility: $\mathcal{V} = |\mathcal{S}(t)|$

It gives the autocorrelation function of the crystal:

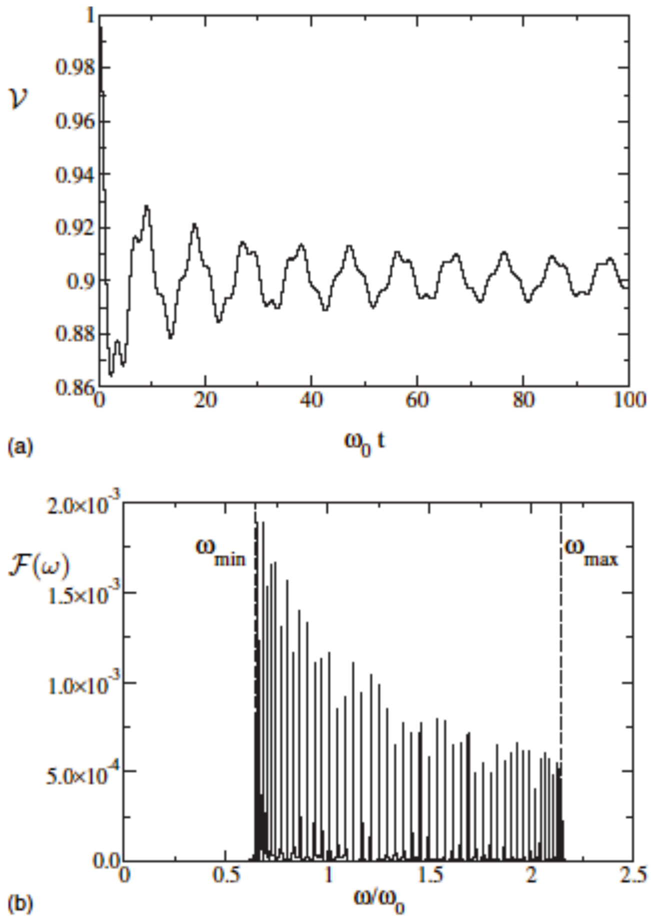
$$\mathcal{V}(t) = \exp\left[-\frac{k_L^2}{2}\mathcal{G}(t)\right]$$

$$\mathcal{G}(t) = \langle[y(t) - y(0)]^2\rangle$$

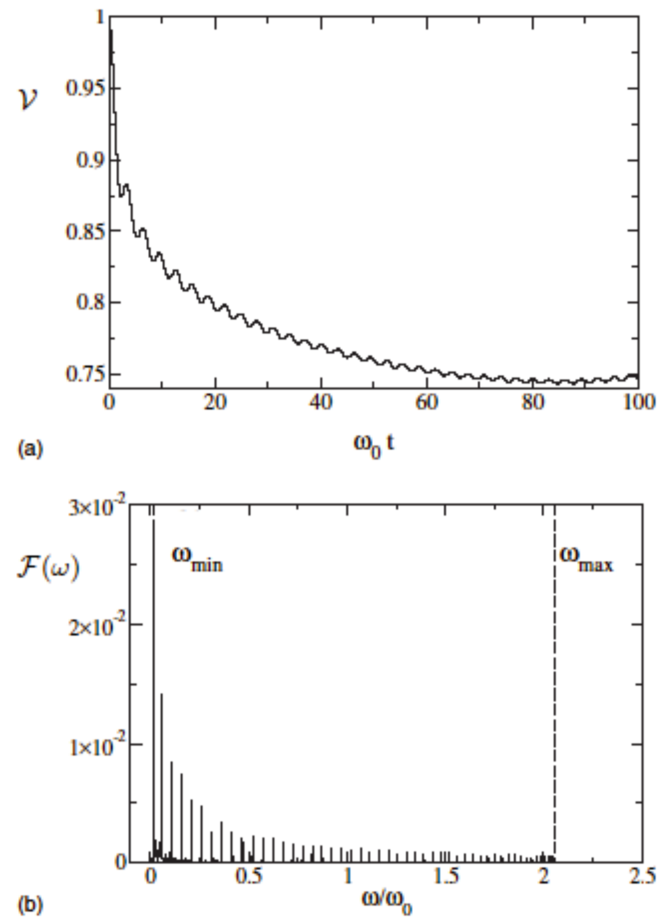
Visibility for an ion chain

$$\mathcal{V} = |\langle \{\alpha_{k\sigma}\} | \{\alpha_{k\sigma}(t)\}\rangle| = \exp[-A(t)] \qquad A(t) = 2\sum_{k\sigma} |\alpha_{k\sigma}|^2 \sin^2 \frac{\omega_y(k)t}{2}$$

Away from the critical point:



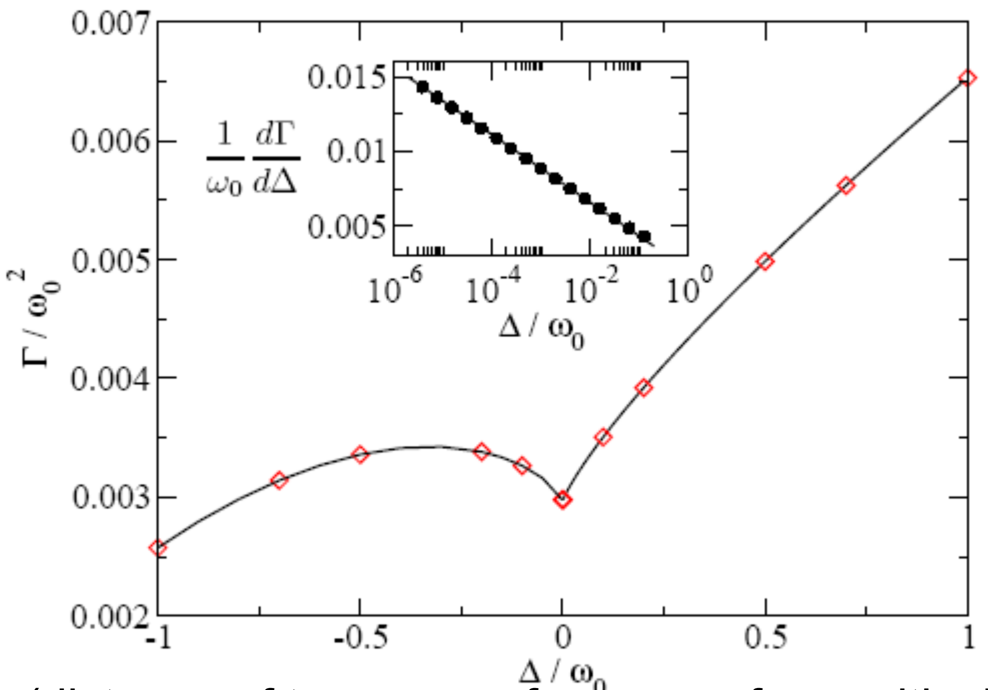
Close to the critical point:



Visibility at the critical point

$$\mathcal{V} = |\langle\{\alpha_{k\sigma}\}|\{\alpha_{k\sigma}(t)\}\rangle| = \exp[-A(t)]$$

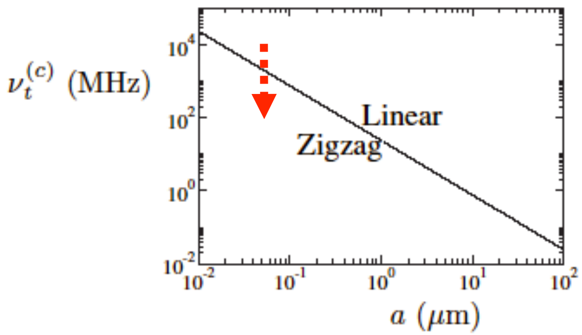
$$A(t) = 2\sum_{k\sigma}|\alpha_{k\sigma}|^2 \sin^2\frac{\omega_y(k)t}{2} \simeq \Gamma t^2$$ (short elapsed times)



(distance of transverse frequency from critical value)

Testing criticality with quenches

Slow quench across the phase transition



$$\nu_t = \sqrt{\nu_t^{(c)2} + \epsilon(t)}$$

$$\epsilon(t) = -\delta_0 \frac{t}{\tau_Q}$$

$$\tau \sim |\epsilon|^{-z\nu}$$

Kibble-Zurek mechanism:

Adiabatic regime: quench time << relaxation time

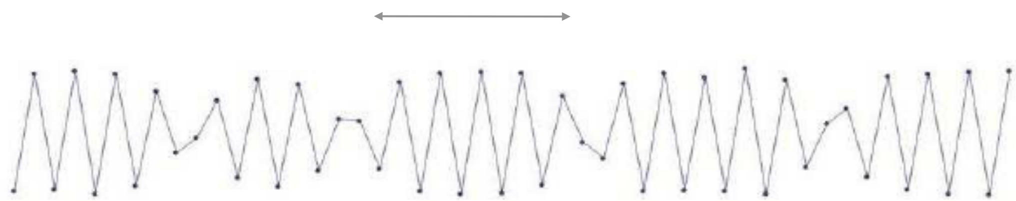
Impulse regime: quench time >> relaxation time

Defects are formed when quench time = relaxation time

Density of defects after the quench

Scaling of the density of defects as a function of the quench rate:

Use $\tau \sim |\epsilon|^{-z\nu}$ in $\hat{\xi} \sim |\hat{\epsilon}|^{-\nu}$ (domain size at freeze-out)

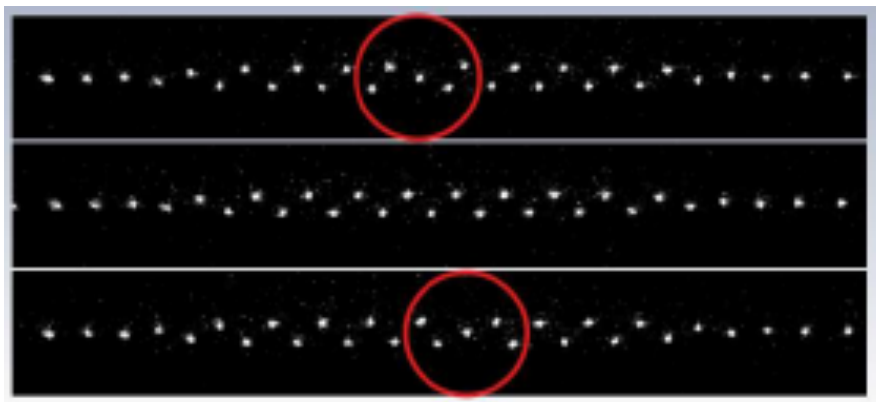


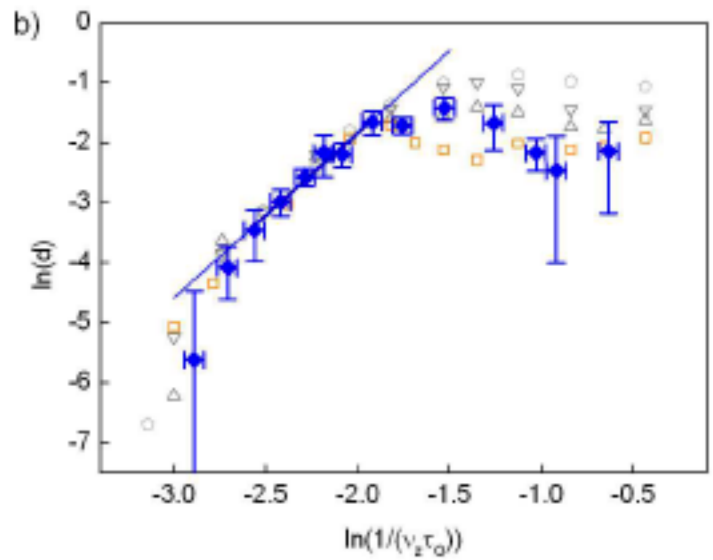
$$\hat{\xi} \sim |\hat{\epsilon}|^{-\nu} \sim \tau_Q^{\nu/(1+z\nu)}$$

it depends on the critical exponents
it provides a way to measure the critical behaviour

Experimental quenches

$$\xi \sim \tau_Q^{4/3}$$





T. Mehlstäubler and coworkers (PTB)

Evidence of the classical criticality
(in agreement with KZ in inhomogeneous systems)

See also: Schätz (Freiburg) and Schmidt-Kaler (Mainz)

Seeing quantum criticality?

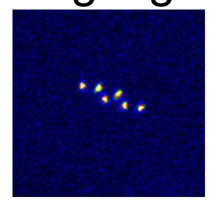
$$\hat{\xi} \sim |\hat{\epsilon}|^{-\nu} \sim \tau_Q^{\nu/(1+z\nu)}$$

ν_t

linear

disordered phase

zigzag



classical (mean-field) exponents

$$\nu = 1/2$$

$$z = 1$$

$$\xi \sim \tau_Q^{1/3}$$

quantum critical region

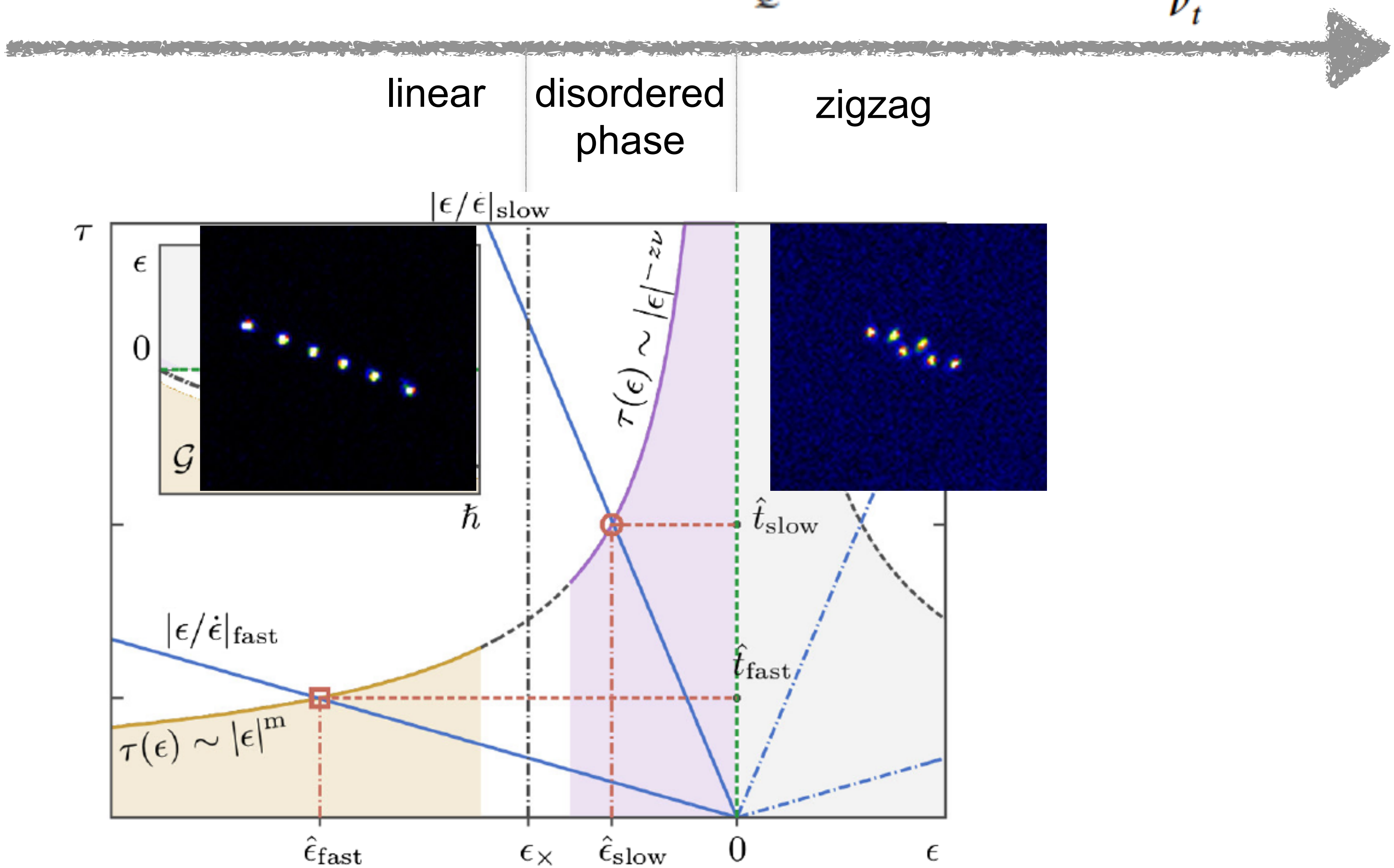
$$\nu = 1$$

$$z = 1$$

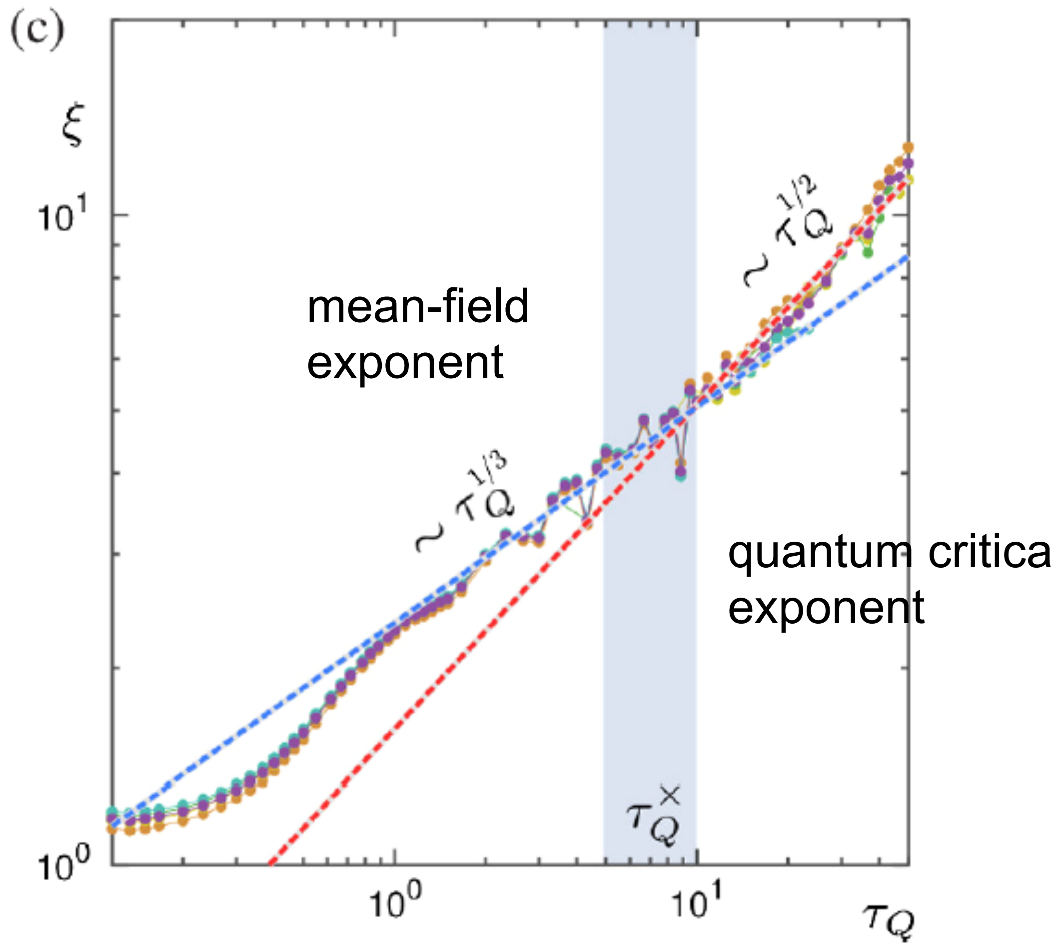
$$\hat{\xi} \sim \tau_Q^{1/2}$$

Seeing quantum criticality?

$$\hat{\xi} \sim |\hat{\epsilon}|^{-\nu} \sim \tau_Q^{\nu/(1+z\nu)}$$



Seeing quantum criticality



(DMRG by Pietro Silvi, Uni Ulm)

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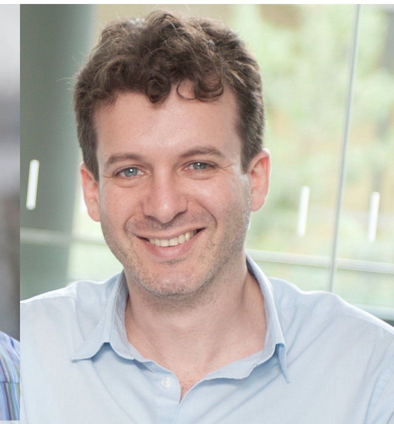

E. Shimshoni

D. Podolsky

S. Montangero

P. Silvi

A. del Campo

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