Quantum Simulation with Continuous-Variable Systems

or: of Feynman, Bennett, Nicholson, the hand of God, and what they have in common

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Varenna, 24 July 2015

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Belfast

Titanic

"...the world commemorates the tragedy; only we can celebrate the triumph..."

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On the shoulders of Belfast's giants

John Stuart Bell

Joseph Larmor

Born in Belfast in 1824

Lord Kelvin

Belfast, Botanic Gardens

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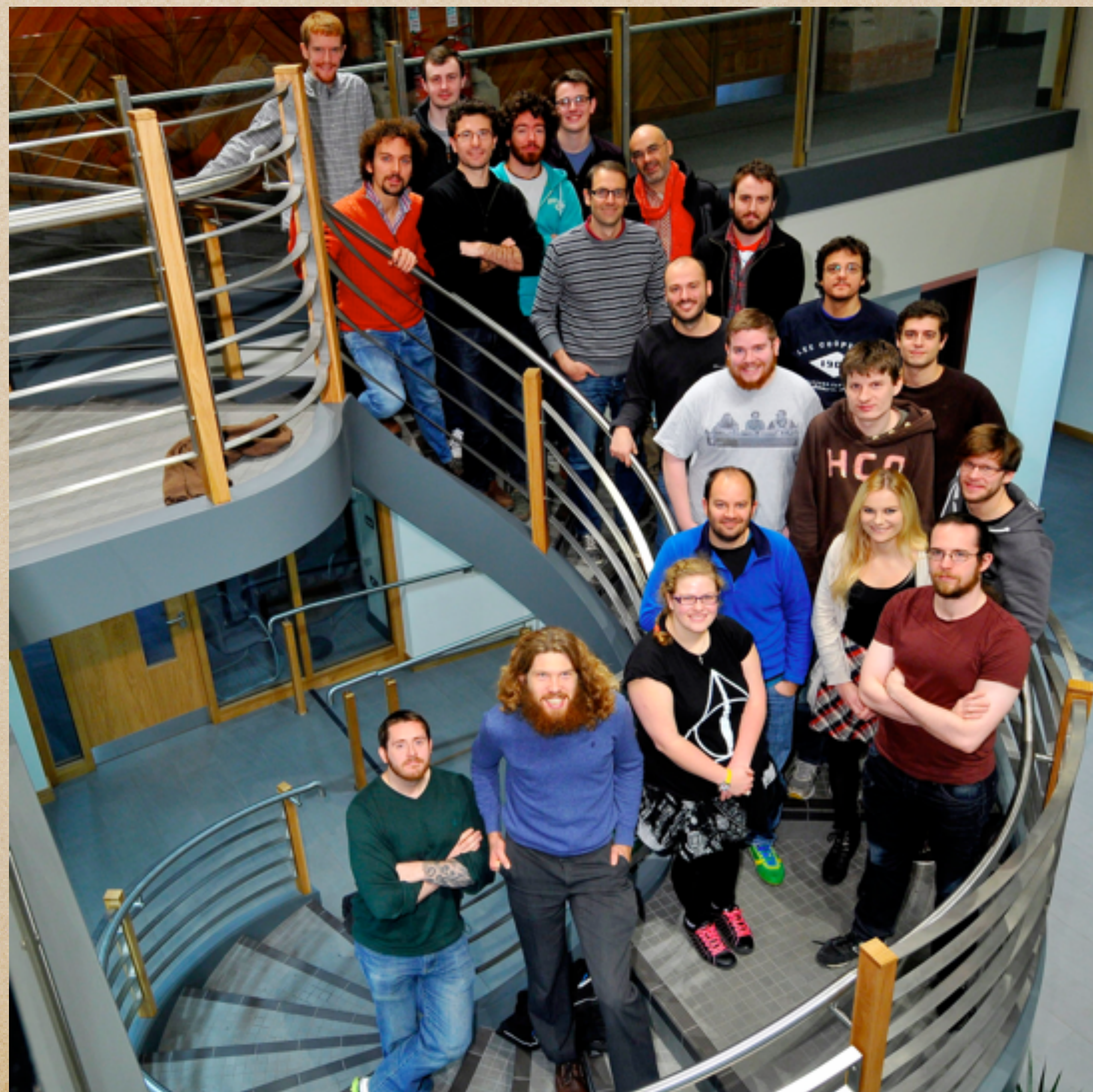
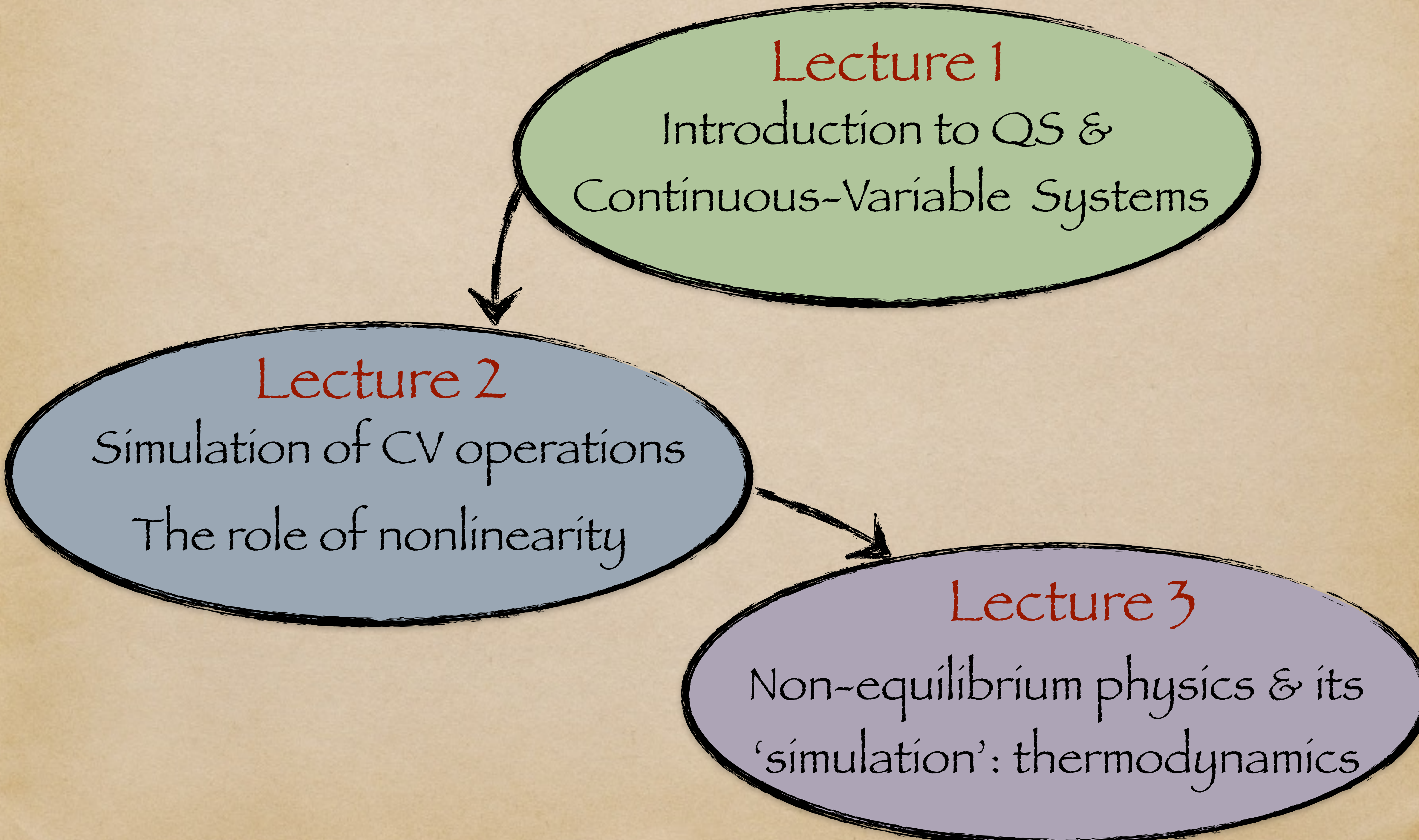
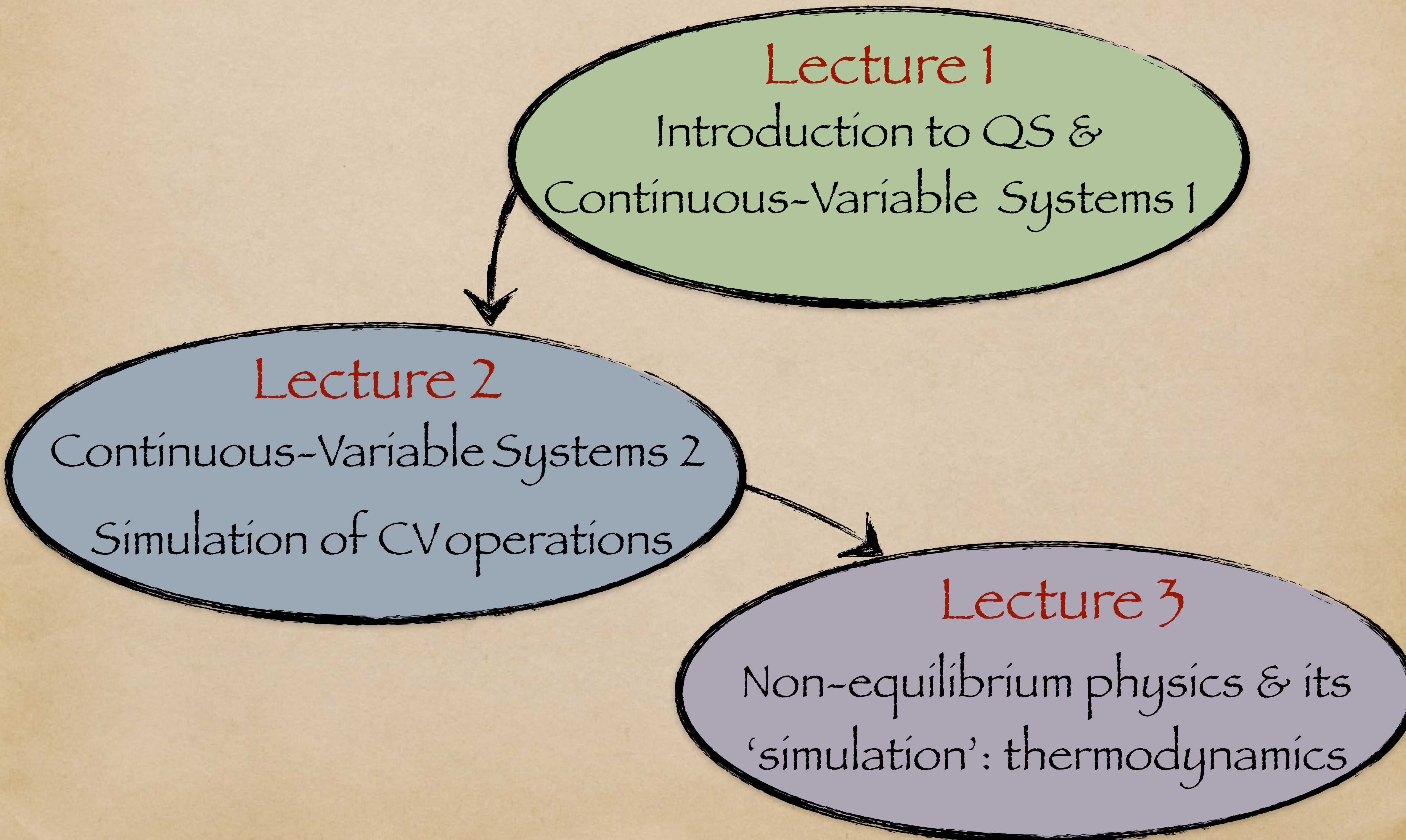
The Belfast crew



Plan of the lectures



Being realistic..



Plan of the lectures

Lecture 1

General (incomplete and personal) introduction to QS

General features of digital & analog QS

Introduction to Continuous-Variable Systems (CVSs)

He knew best

R. Feynman

...trying to find a computer simulation of physics, seems to me to be an excellent program to follow out...and I'm not happy with all the analyses that go with just the classical theory, because *nature isn’t classical*, dammit, and if you want to make a simulation of nature, you'd better *make it quantum mechanical*, and by golly it's a wonderful problem because it doesn't look so easy.”

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Premises & warnings

Quantum Simulator: a controllable quantum system used to reproduce/emulate other quantum systems or their properties.

Not necessarily we need to reproduce the whole system to be simulated!

It's the statistics/evolution that matters!

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Simulation?
Working by examples

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Premises & warnings

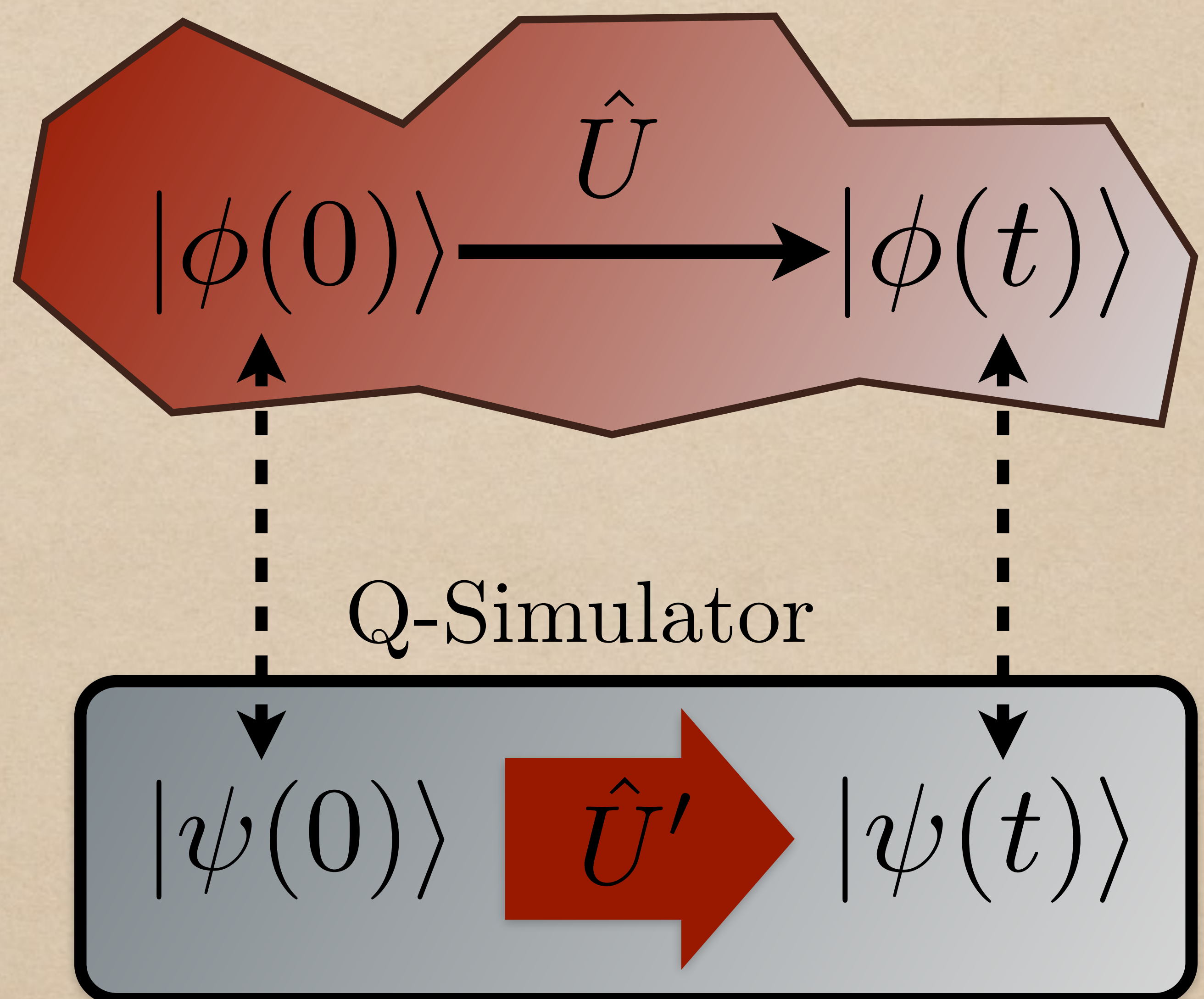
Quantum Simulation

Digital

Analog

S. Lloyd, "Universal quantum simulators", Science 273, 1073 (1996).

General picture



The mapping is key to the success of the simulation

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All steps need to be accurate

The mapping is key to the success of the simulation

Charlie Bennett

$\xrightarrow{\hat{U}}$

PHYSICAL REVIEW LETTERS

VOLUME 70 | 29 MARCH 1993 | NUMBER 13

Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels

Charles H. Bennett,[1] Gilles Brassard,[2] Claude Crépeau,[2],[3] Richard Jozsa,[2] Asher Peres,[4] and William K. Wootters[5]

$\xrightarrow{\hat{U}'}$

Jack Nicholson

The shining (Stanley Kubrik)

The whole simulation needs to be faithful

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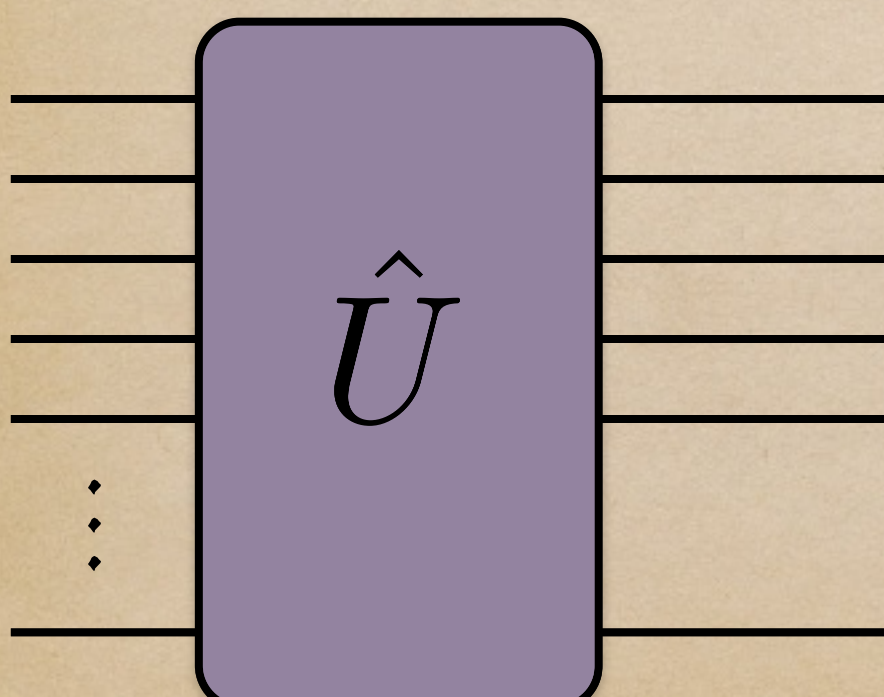
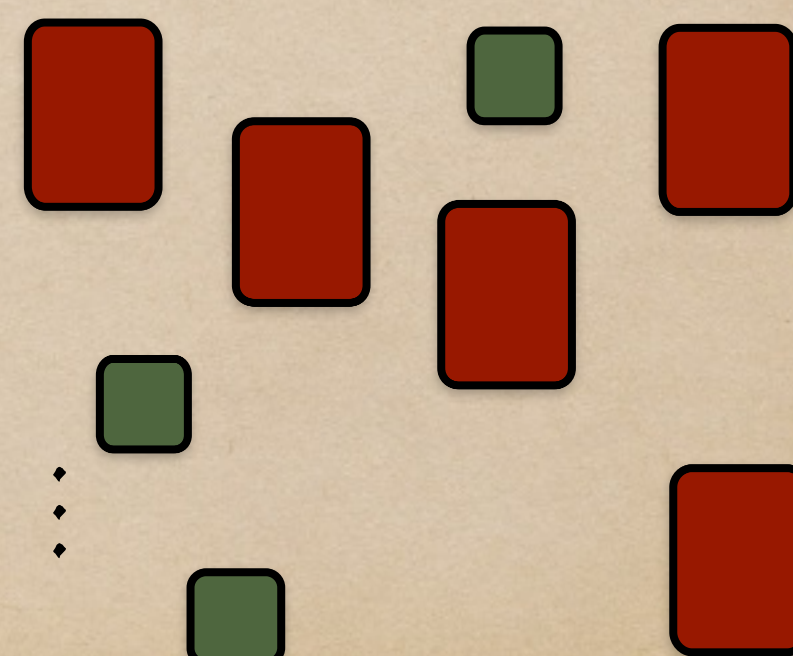
Digital quantum simulation (DQS)

A unitary realisation of a transformation based on single- & two-qubit gates

DQS is universal





He figured out everything...

...The rule of simulation that I would like to have is that the number of computer elements required to simulate a large physical system is only to be proportional to the space-time volume of the physical system. I don’t want to have an explosion.

.”

R. Feynman, “Simulating physics with computers,”

Int. J. Theor. Phys. 21, 467 (1982)

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DQS: Dynamics

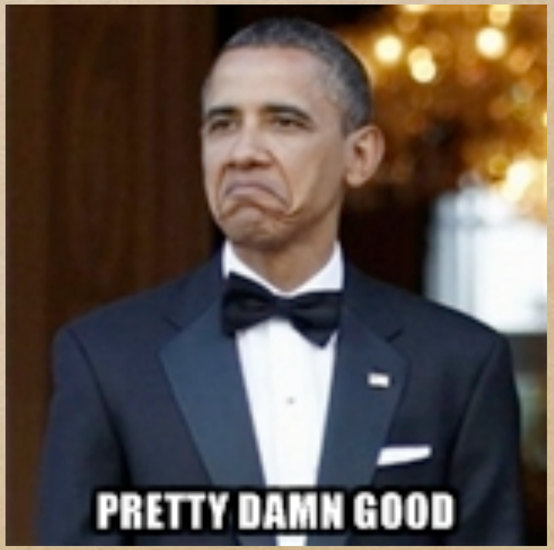
How about efficiency?

Not any unitary can be efficiently simulated

Not any Hamiltonian/evolutions can be efficiently simulated

All finite-dimensional local Hamiltonians can be simulated efficiently!

All the Hamiltonians that can be mapped into local models can be efficiently simulated



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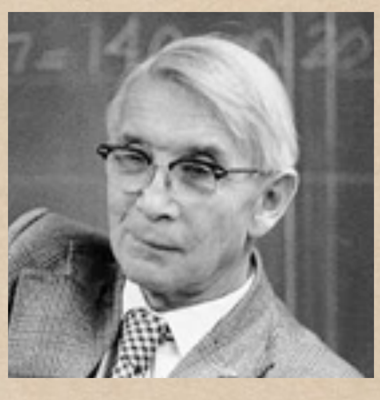
DQS: Dynamics

$$H = \sum_{l=1}^{M} H_l$$



Ernst Ising

$$H_I = J \sum_i \sigma_i^x \otimes \sigma_{i+1}^x$$

$$[H_l, H_{l'}] = 0$$

$$U = \prod_l \exp\{-i\hbar H_l t\}$$

The simulation is straightforward

$$[H_l, H_{l'}] \neq 0$$

DQS: Dynamics

$[H_l, H_{l'}] \neq 0$

$$U = (\exp\{-i\hbar H \Delta t\})^{t/\Delta t}$$

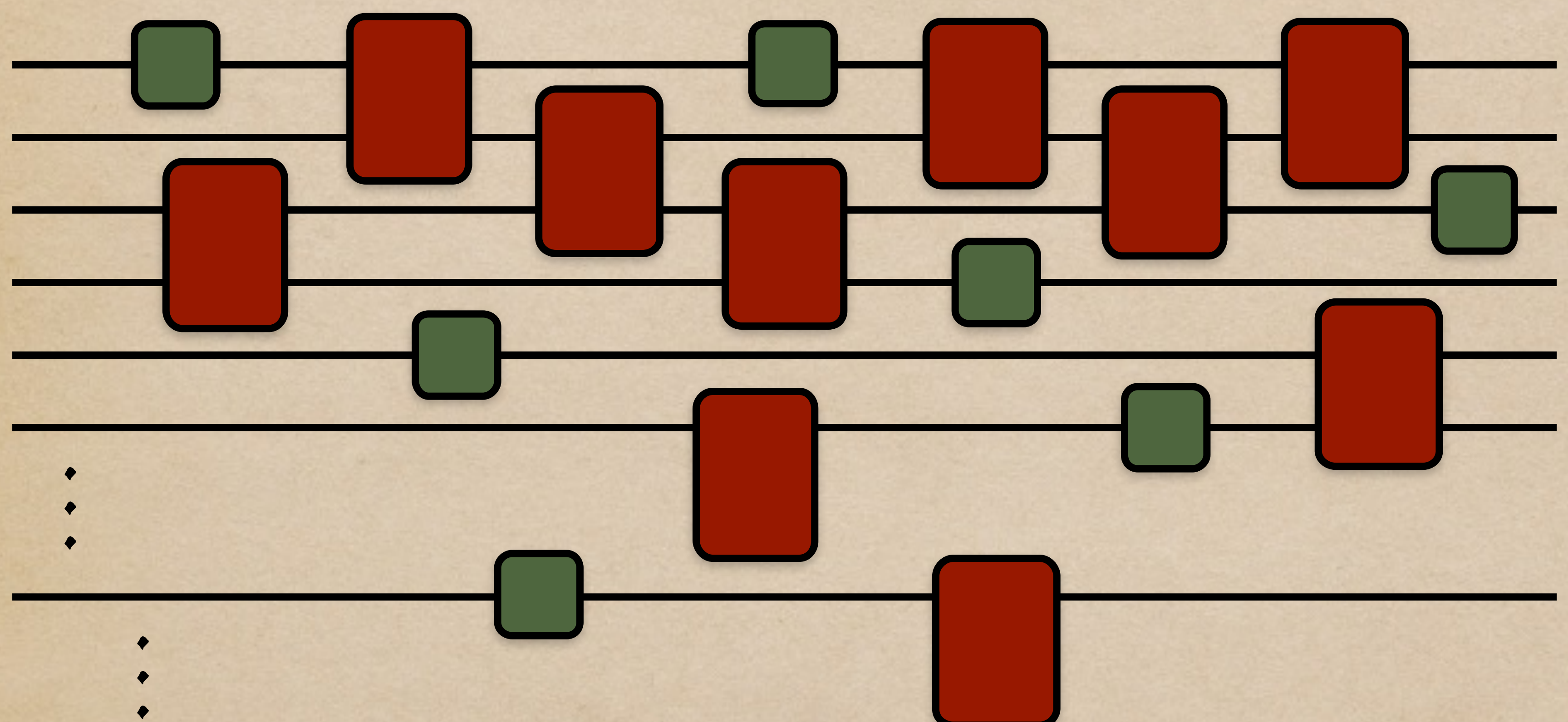
$$e^{A+B} = \lim_{n\to\infty} \left(e^{\frac{A}{n}} e^{\frac{B}{n}}\right)^n$$ Suzuki-Trotter expansion

$$e^{\delta(A+B)} = \lim_{\delta\to 0} \left(e^{\delta A} e^{\delta B}\right) + \mathcal{O}(\delta^2)$$

$$U(\Delta t) = e^{-i\hbar \sum_l H_l \Delta t} = \prod_l e^{-i\hbar H_l \Delta t} + \mathcal{O}((\Delta t)^2)$$

$$\approx \prod_l \exp\{-i\hbar H_l \Delta t\} \quad \text{as } \Delta t \to 0$$

DQS: Dynamics



Higher-order expansions typically help reducing complexity

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DQS: Dynamics

$$U_{\text{tot}} = e^{iH_4\delta t}e^{iH_3\delta t}e^{iH_2\delta t}e^{iH_1\delta t}$$

Taylor-expand up to 2nd order in δt

$$U_{\text{tot}} = \mathbb{1} + i\delta t\sum_{j=1}^{4} H_j - \frac{\delta t^2}{2}\left(\sum_{j=1}^{4} H_j^2 + 2\sum_{l=2}^{4}\sum_{j=1}^{l-1} H_l H_j\right) + O(\delta t^3)$$

Now take $H_4 = -H_2, \qquad H_3 = -H_1$

$$\begin{aligned} U_{\text{tot}} &= \mathbb{1} + \delta t^2(H_1H_2 - H_2H_1) + O(\delta_t^3) \\ &= e^{-i2\delta t^2 \times \boxed{\underbrace{(i/2)[H_1,H_2]}_{= H_{\text{eff}}}}} + O(\delta t^3) \end{aligned}$$

DQS: Dynamics

$$U_{\text{tot}} = e^{-i2\delta t^2 \times (i/2)[H_1, H_2]} + O(\delta t^3)$$

Suppose availability of only 2-body Hamiltonians (cheap!)

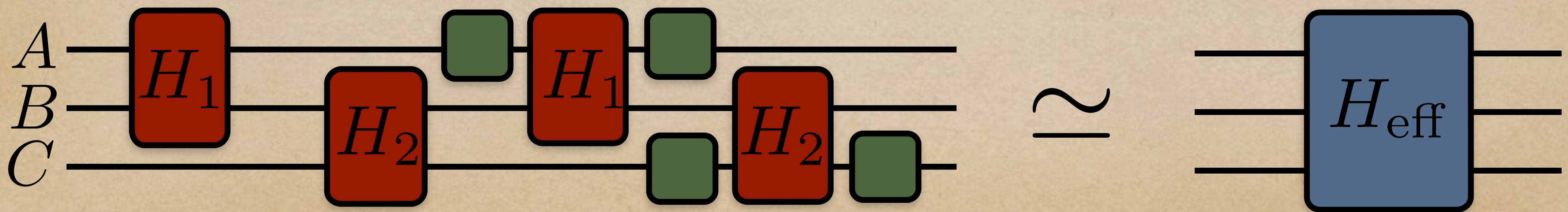
Is it possible to achieve effective 3-body couplings?!!

$$H_1 = \sigma_z^{(A)}\sigma_x^{(B)}$$

$$H_2 = \sigma_y^{(B)}\sigma_z^{(C)}$$

$H_{\text{eff}} = \sigma_z^{(A)}\sigma_z^{(B)}\sigma_z^{(C)}$ for a time $\delta t' = 2\delta t^2$

Error of the order of $O(\delta t'^{3/2})$



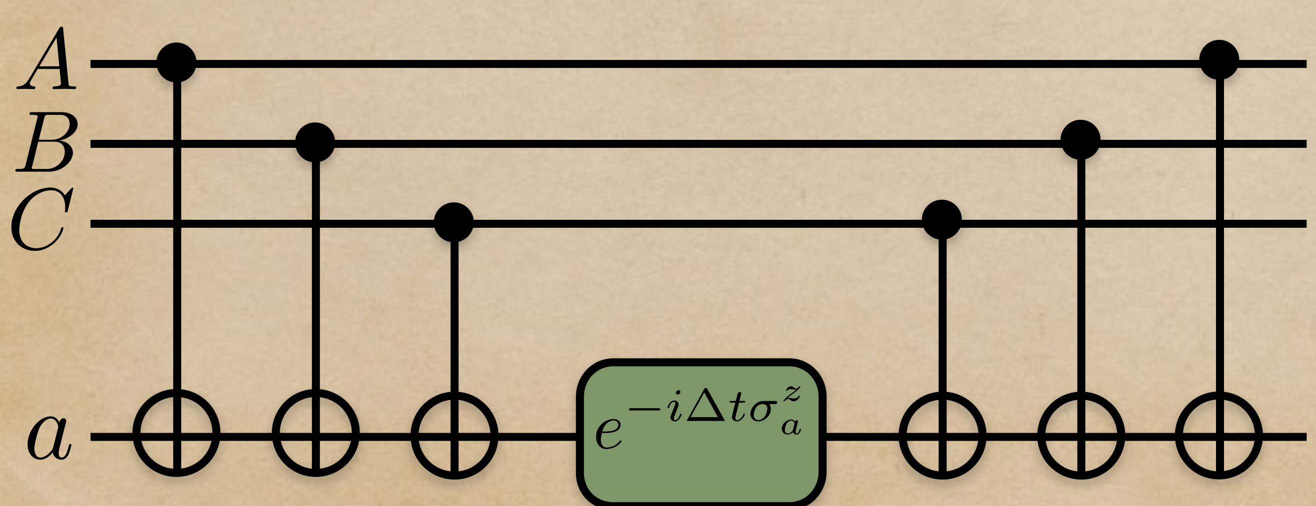
DQS: Dynamics

General case: two n-body interactions can simulate a $m = (2n-1)$-body interaction

$$U_{\text{tot}} = \mathbb{1} + iO(\delta t^{m-1})H_{\text{eff}} + O(\delta t^m) = \mathbb{1} + i\delta t_m H_{\text{eff}} + O(\delta t_m^{m/(m-1)})$$
$$\approx e^{i\delta t_m H_{\text{eff}}}$$

time $\delta t_m = O(\delta t^{m-1})$ error $O(\delta t_m^{m/(m-1)})$

$H_{\text{eff}} = \sigma_z^{(A)}\sigma_z^{(B)}\sigma_z^{(C)}$



$\equiv$

Nielsen & Chuang

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DQS: Measurement

Quantum state tomography

Quantum process tomography

Resources scale exponentially!

Often, it is asking too much

Key, statistical information (spectra, correlators) are often sufficient

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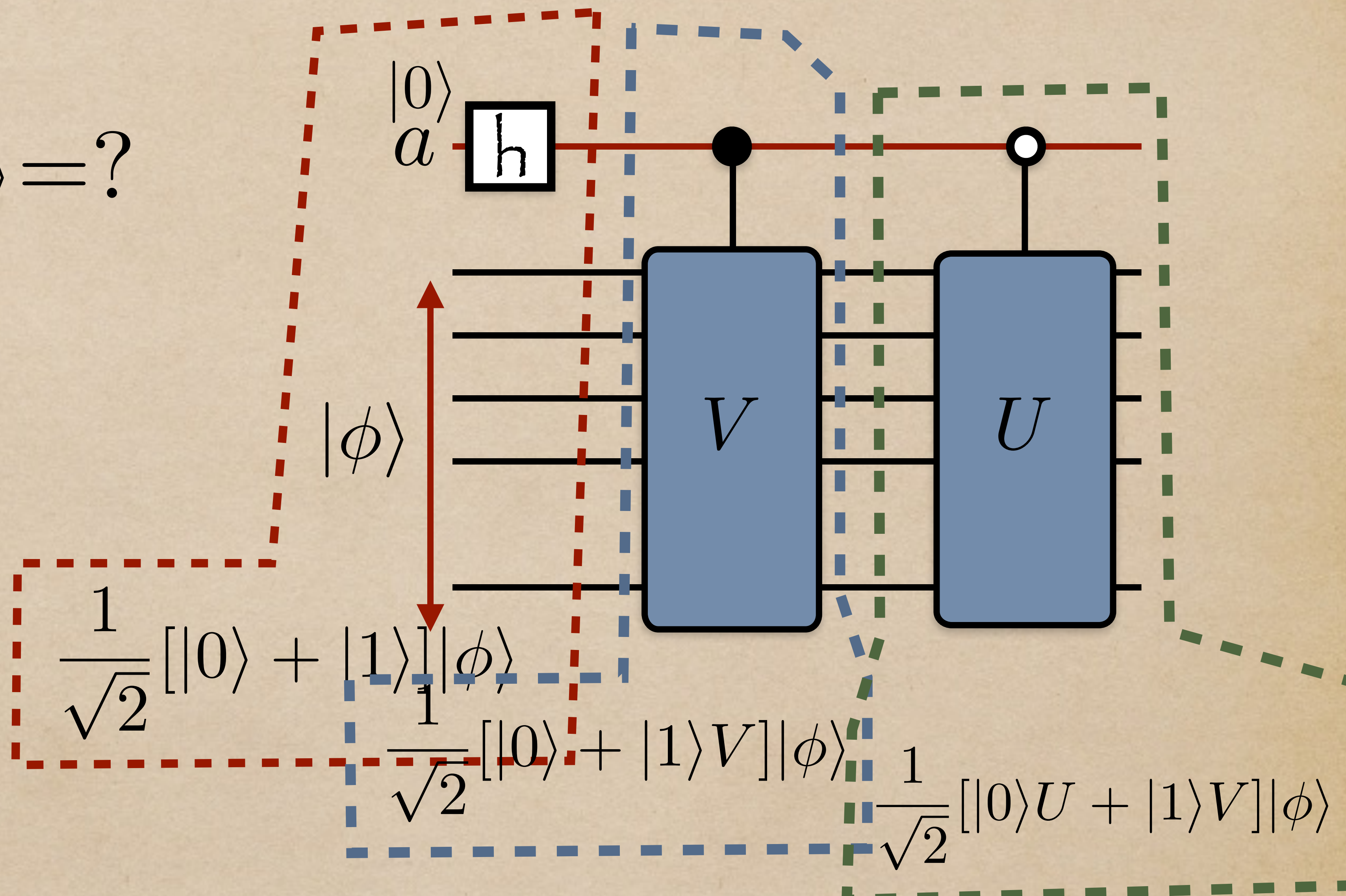
DQS: Measurement

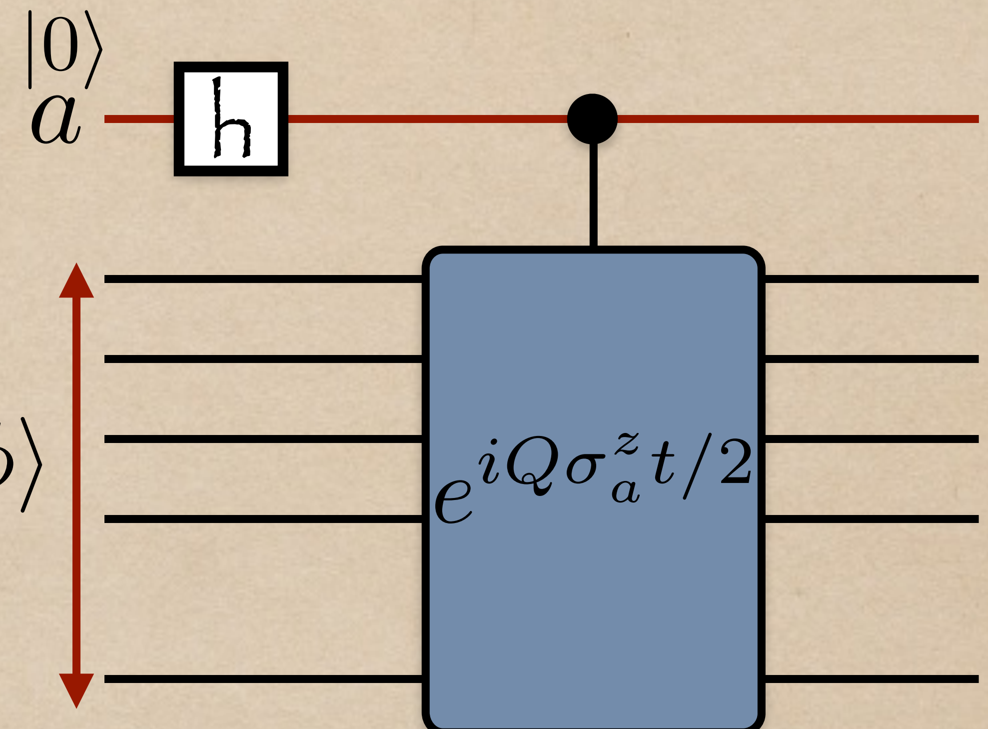
$\langle U^\dagger V \rangle = ?$



$$\rho_a = \frac{1}{2}[|0\rangle\langle 0| + |1\rangle\langle 1| + (|1\rangle\langle 0| \boxed{\langle\phi|U^\dagger V|\phi\rangle} + h.c.)]$$

DQS: Measurement

Eig[Q]=?



$$\sum_{n=0}^{\mathcal{L}} \gamma_n \, |\Psi_n\rangle = |\phi\rangle$$

$$2\langle \sigma_a^z \rangle = \langle \phi | e^{-iQt} | \phi \rangle \; = \sum_{n=0}^{\mathcal{L}} |\gamma_n|^2 \, e^{-i\lambda_n t}$$

$$\xrightarrow{\text{FFT}} \sum_{n=0}^{\mathcal{L}} 2\pi |\gamma_n|^2 \delta(\lambda - \lambda_n)$$

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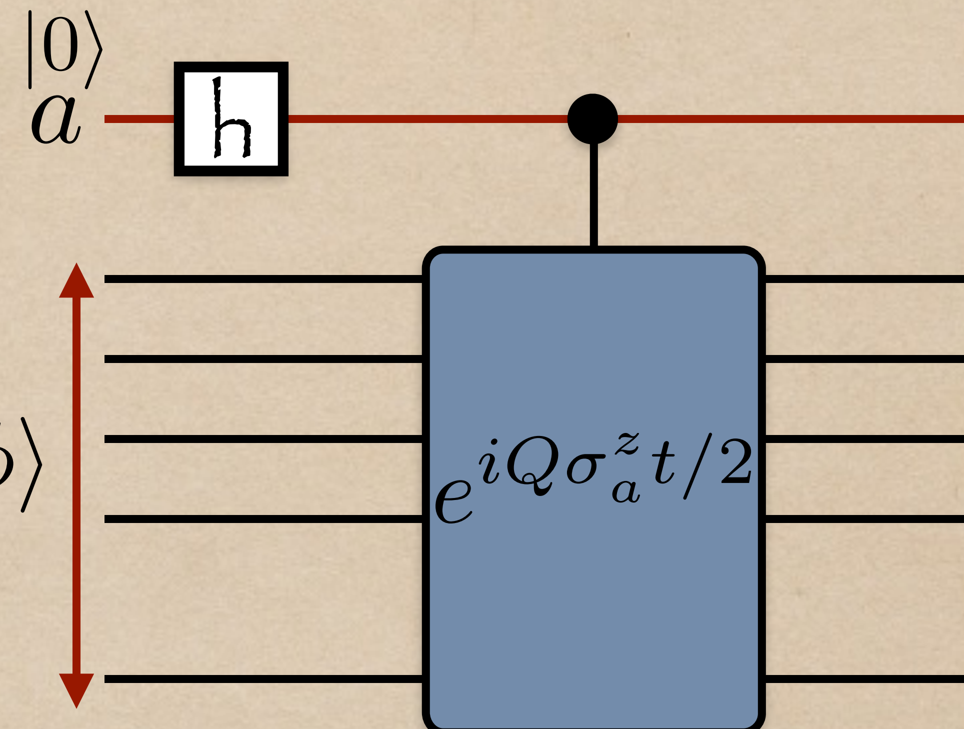
DQS: Measurement

$\mathrm{Eig}[Q] = ?$

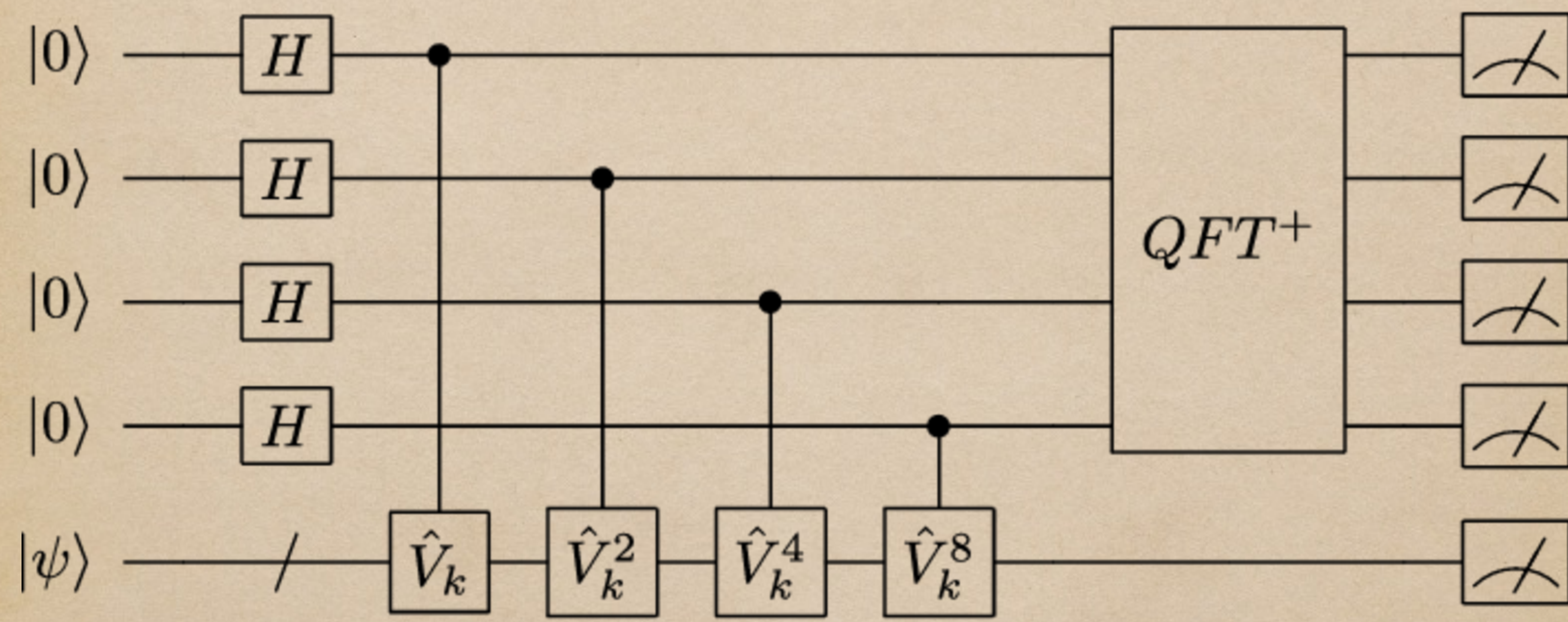
$$\sum_{n=0}^{\mathcal{L}} \gamma_n \, |\Psi_n\rangle = |\phi\rangle$$



$$2\langle \sigma_a^z \rangle = \langle \phi | e^{-iQt} | \phi \rangle \; = \sum_{n=0}^{\mathcal{L}} |\gamma_n|^2 \, e^{-i\lambda_n t}$$

$$\xrightarrow{\text{FFT}} \sum_{n=0}^{\mathcal{L}} 2\pi |\gamma_n|^2 \delta(\lambda - \lambda_n)$$

DQS: Measurement



Calculation of energies of a molecule

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Analog quantum simulation (AQS)

Analog quantum simulation requires a quantum system to mimic (emulate) another

$$H_{\text{sys}} \longleftrightarrow H_{\text{sim}}$$

Controllable to some extent

the problem

the model

the emulator?!?!

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Analog quantum simulation (AQS)

Analog quantum simulation requires a quantum system to mimic (emulate) another

$$H_{\mathrm{sys}} \longleftrightarrow H_{\mathrm{sim}}$$

Controllable to some extent

The emulator may only reproduce partially the dynamics of the system

AQS is well suited to emulate an effective many-body model

A controllable "toy-model" of the system is used to reproduce the property of interest, e.g. the dynamics or ground state.

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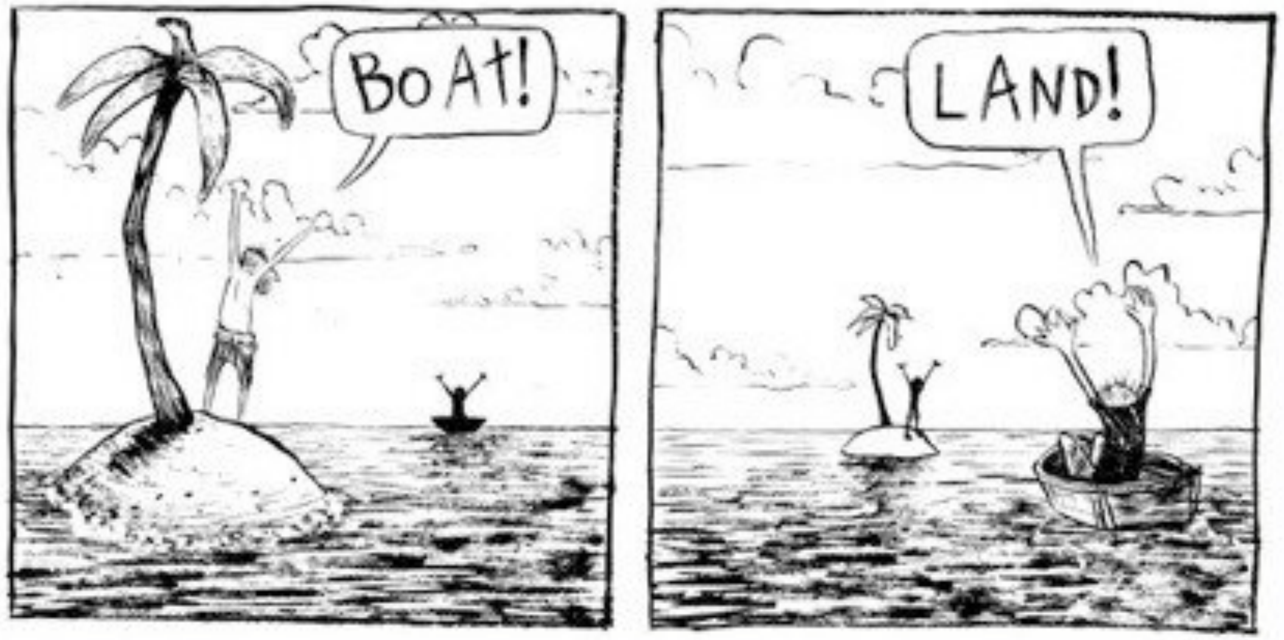
Analog quantum simulation (AQS)

AQS easier than DQS?

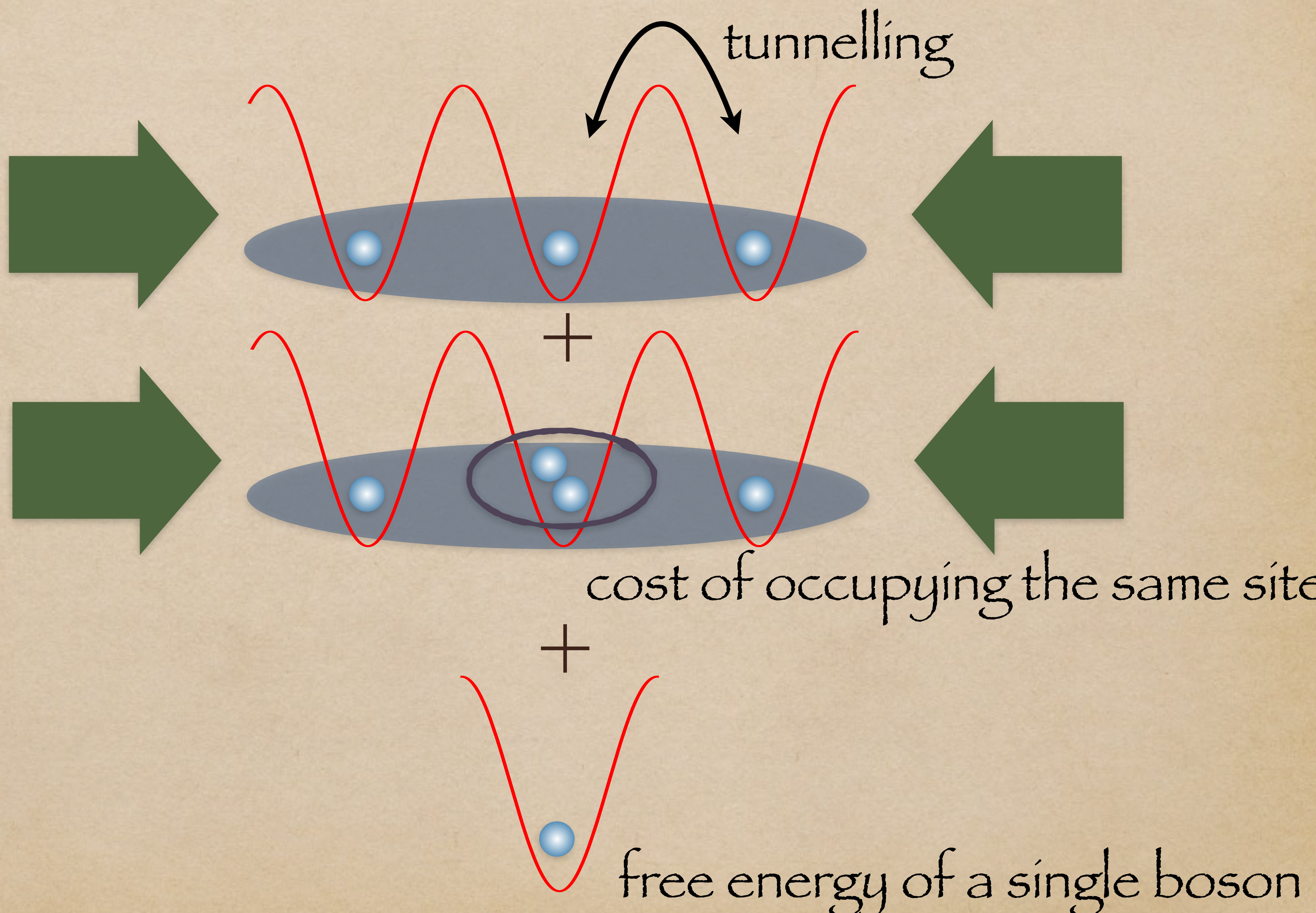
Not necessarily!

Finding the right map!

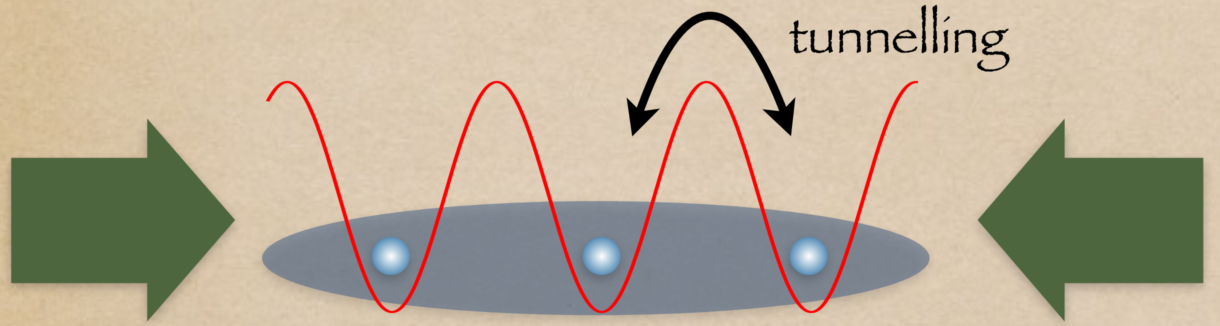


Perspective...

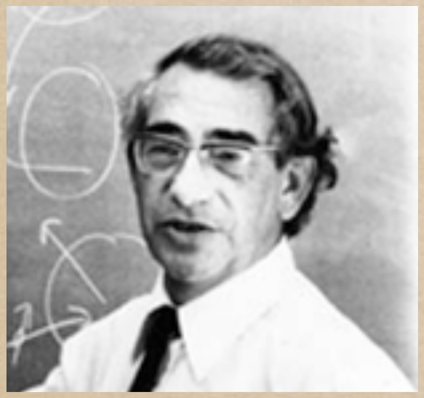
Analog quantum simulation (AQS)



AQS of BH model



$$H_{\rm sim} = -J \sum_{i,j} \hat{a}_i^\dagger \hat{a}_j + \sum_i \epsilon_i \hat{n}_i + \frac{1}{2} U \sum_i \hat{n}_i(\hat{n}_i - 1)$$



John Hubbard

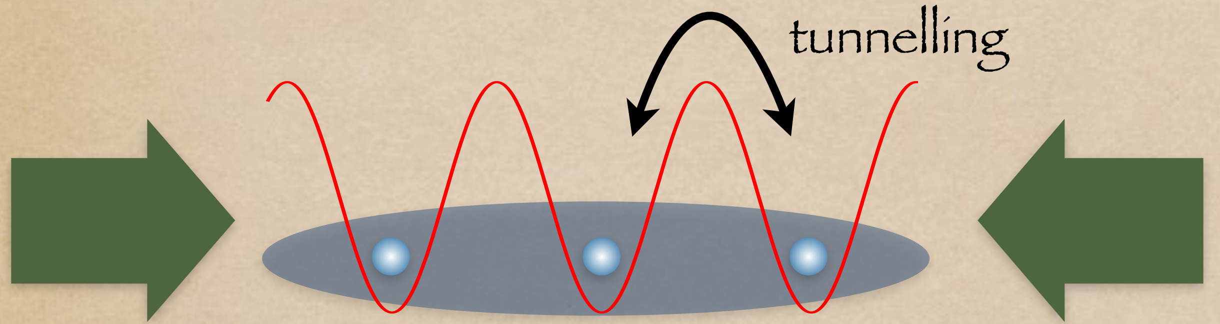
$+$

Satyendra N.Bose

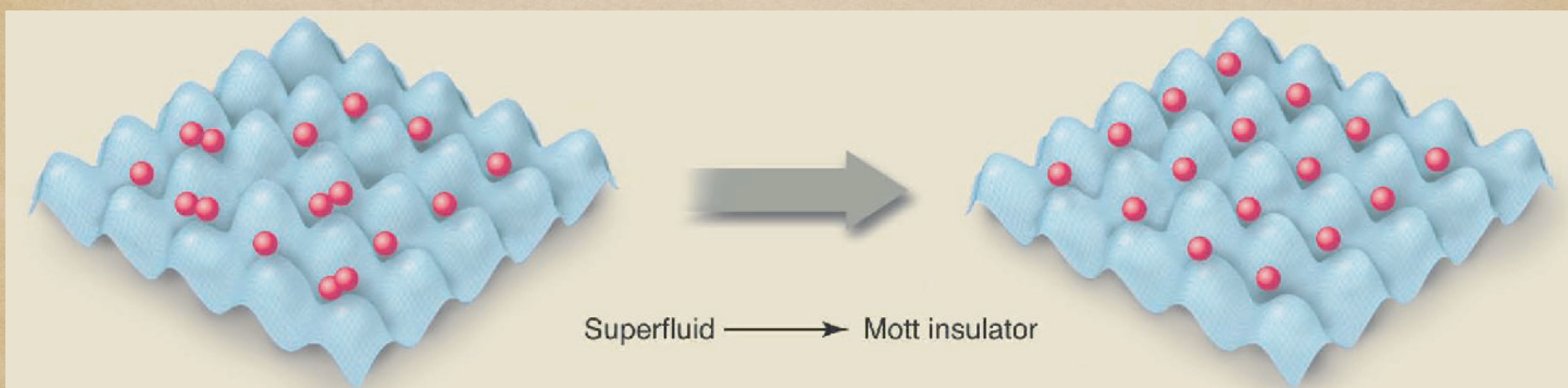
$$H_h = -t \sum_i (c_i^\dagger c_{i+1} + h.c.) + U \sum_i (c_i^\dagger c_i^\dagger)(c_i^\dagger c_i^\dagger - 1)$$

$$H_{\rm BH} = -J \sum_{i,j} \hat{b}_i^\dagger \hat{b}_j + \frac{1}{2} U \sum_i \hat{n}_i(\hat{n}_i - 1) - \mu \sum_i \hat{n}_i$$

AQS of BH model



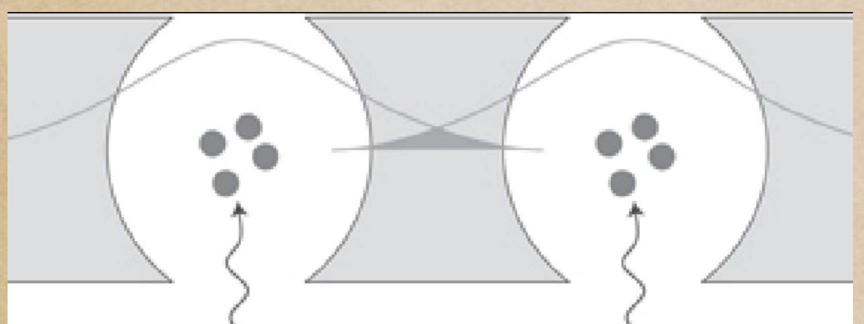
$$H_{\mathrm{sim}} = -J\sum_{i,j}\hat{a}_i^{\dagger}\hat{a}_j + \sum_i \epsilon_i \hat{n}_i + \frac{1}{2}U\sum_i \hat{n}_i(\hat{n}_i - 1)$$



Very important critical effect in many-body physics

Quick overview of some proposals made so far

Condensed matter physics



Bose-Hubbard model via coupled cavity systems

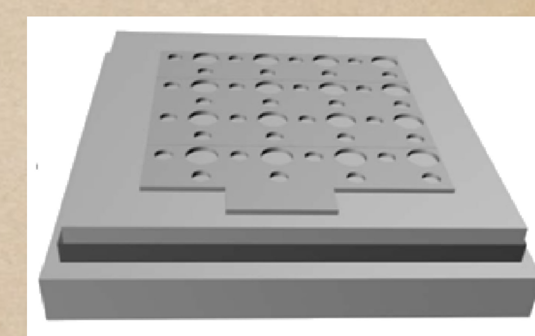
Angelakis et al. PRA 2007 Greentree et al. PRA 2006

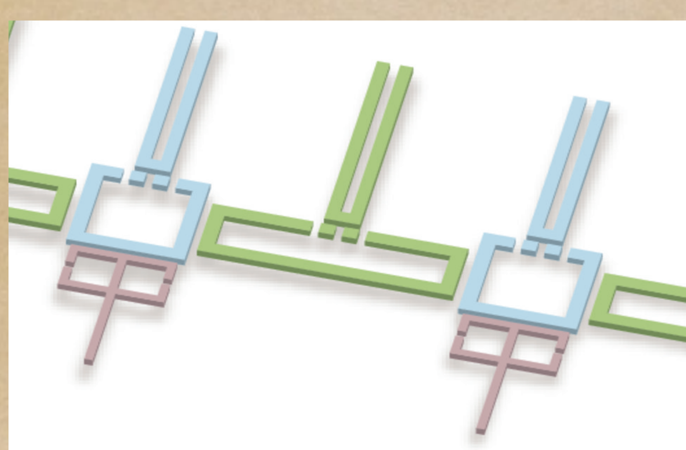
Hartmann et al. Nat. Phys. 2006

Anisotropic Heisenberg model via coupled quantum dots

$$H = \sum_{j=1}^{n} \mu_B g_j(t) B_j(t) \cdot S_j + \sum_{1 \le j < k \le n} J_{jk}(t) S_j \cdot S_k$$

Manousakis et al. JLTP 2002 Byrnes & Yamamoto PRA 2006





Transverse Ising model via coupled superconducting flux qubits

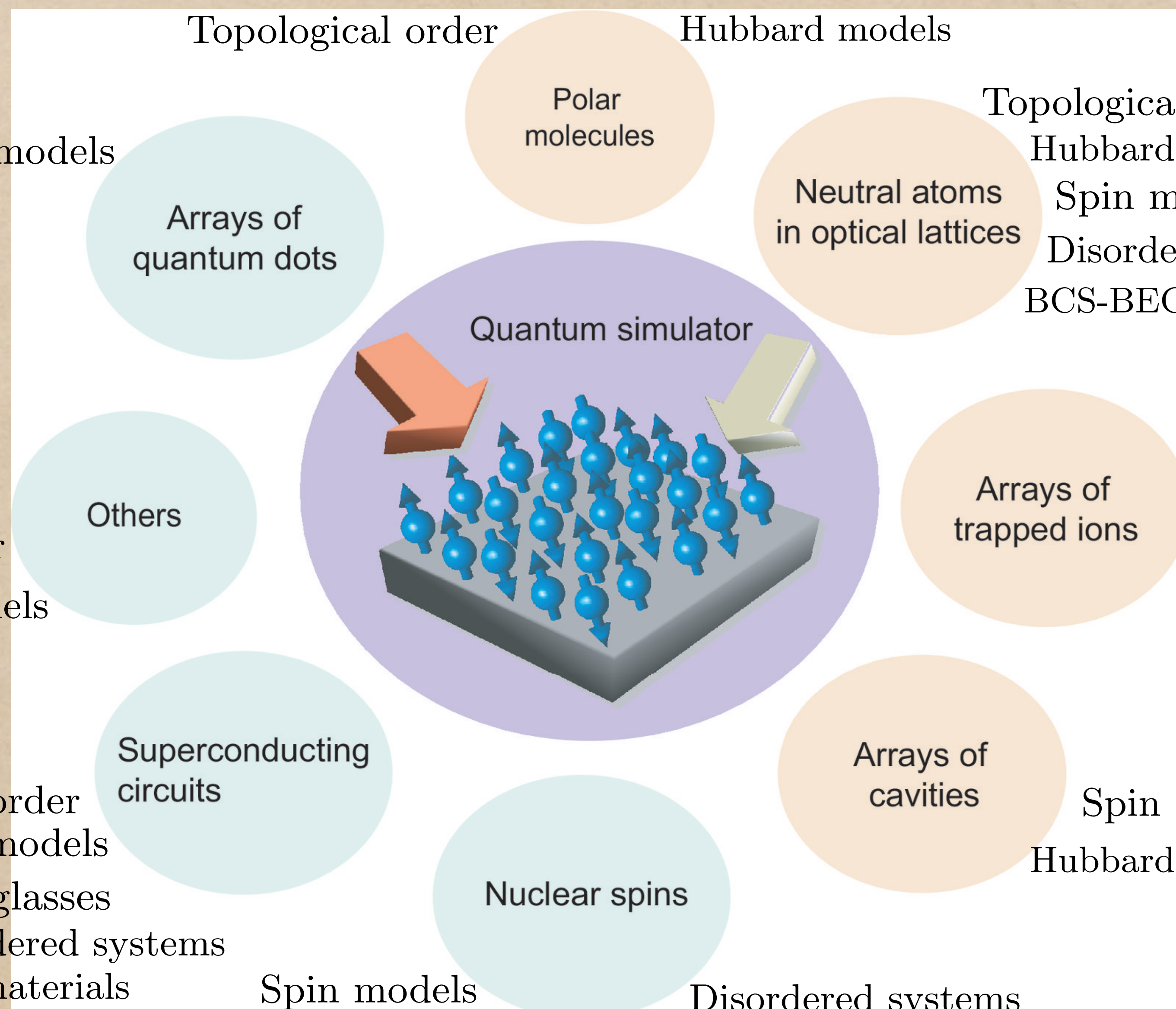
$$H = -\sum_{i=1}^{N} \frac{\Delta_i}{2} \sigma_i^z - \sum_{(i,j)} J_{ij} \sigma_i^x \sigma_j^x$$

Mariantoni et al., Science 2011

Neeley et al., Science 2009

So far....

QS of condensed-matter physics



Topological order
Hubbard models

Hubbard models

Topological order
Hubbard models
Spin models
Disordered systems
BCS-BEC crossover

Hubbard models
Spin models
Disordered systems

Topological order
Spin models

Topological order
Spin models
Spin glasses
Disordered systems
Metamaterials

Spin models
Hubbard models

Spin models
Disordered systems
BCS pairing

I. Georgescu, et al., Rev. Mod. Phys. 2013

How are we doing?

So far so good?

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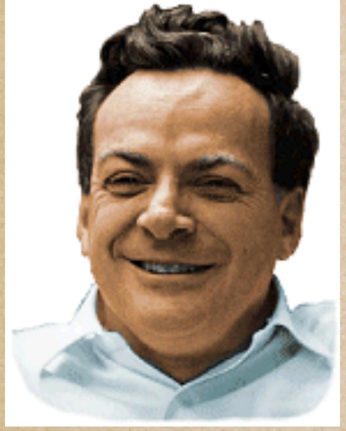
Why bother with CV systems?

Harmonic oscillators are 'natural' information carriers

(can you make an example?)

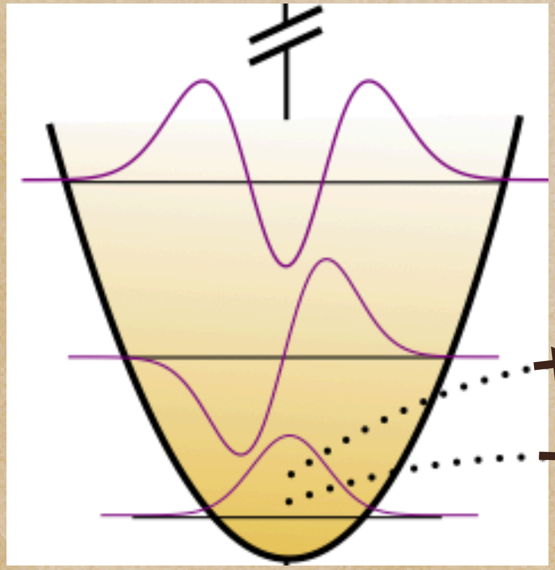


R. Feynman, "There's plenty of room at the bottom", 29 December 1959

↕

"There's plenty of room at the top"

↕



→ Infinite spectrum = more information capacity

→ Somehow, a system of a dual nature

Why bother?

$$\rho_{ho} = I\rho_{ho}I = (\sum_n |n\rangle\langle n|)\rho_{ho}(\sum_m |m\rangle\langle m|) = \sum_{n,m} \rho_{ho}^{nm}|n\rangle\langle m|$$

So, this is like encoding in a qudit (only infinitely large)

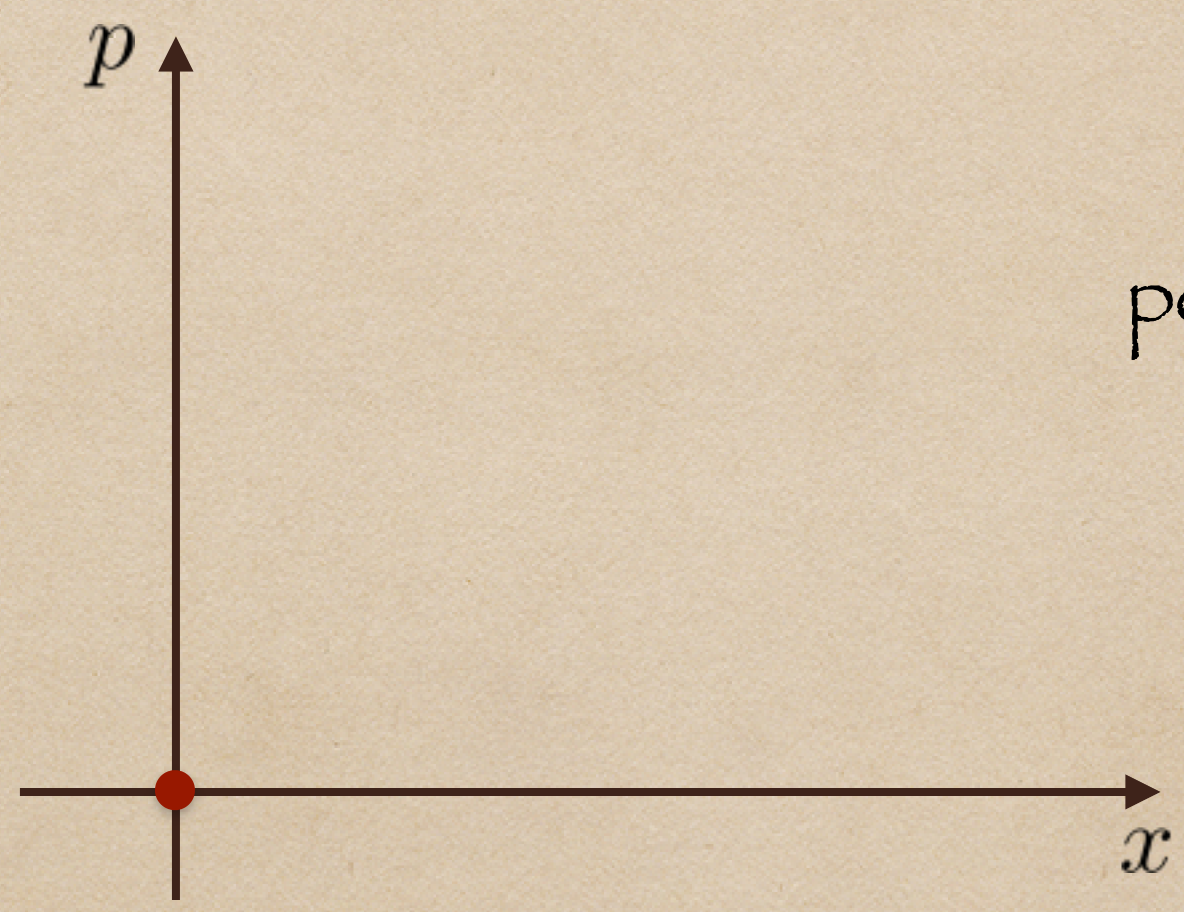
$$\rho_{ho} = I\rho_{ho}I = (\int dx|x\rangle\langle x|)\rho_{ho}(\int dy|y\rangle\langle y|) = \int dxdy\varrho(x,y)|x\rangle\langle y|$$

This is different! We encode in a 'continuous basis'. Why should we?

1. Sometimes, it is just natural
2. Sometimes it is just convenient
3. Why shouldn't we?!

Phase space

Classical



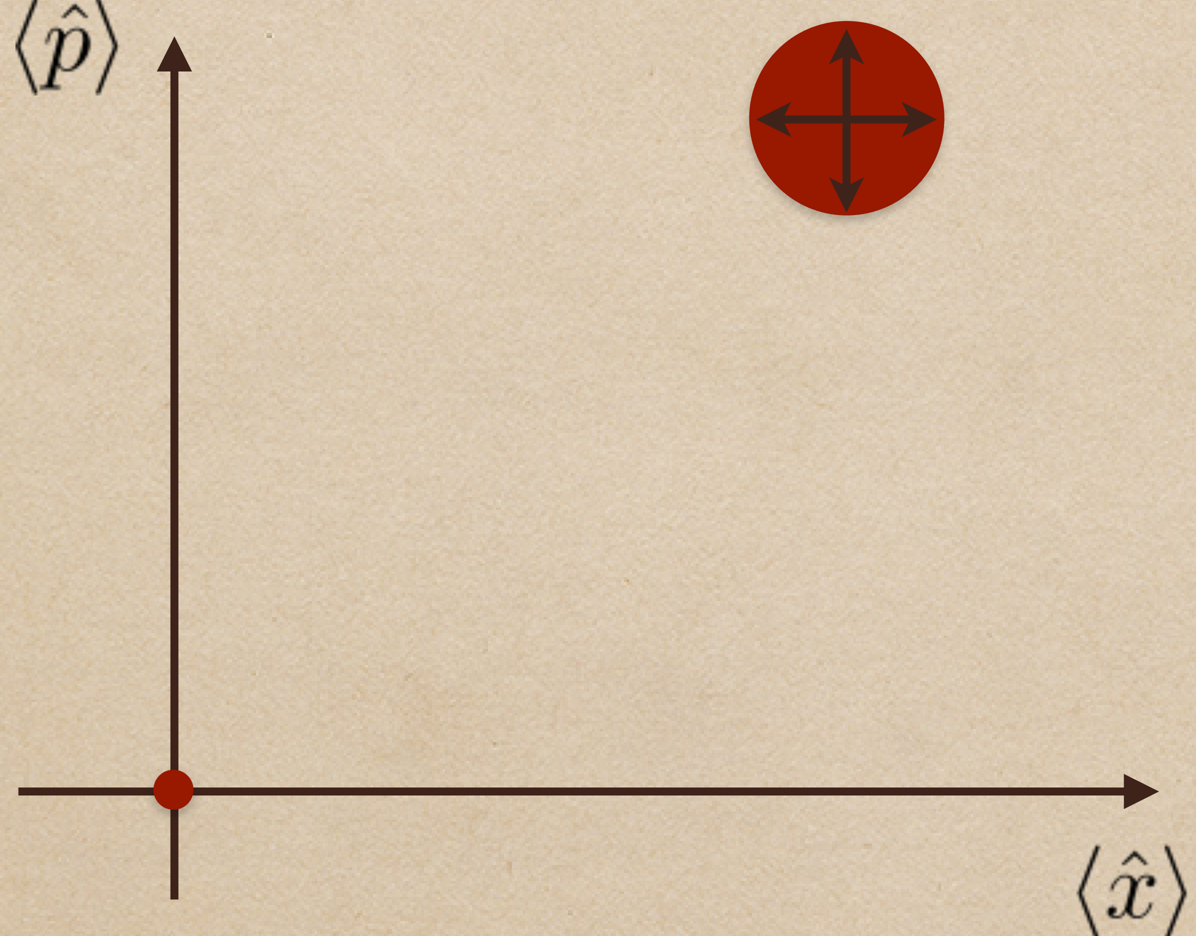
x p

position & momentum

Fully deterministic motion: no uncertainty

Phase space

Quantum



$$\hat{x} = (\hat{a} + \hat{a}^\dagger)/\sqrt{2}$$

$$\hat{p} = i(\hat{a}^\dagger - \hat{a})/\sqrt{2}$$

quadrature operators

$$[\hat{x}, \hat{p}] = i$$

$$(\Delta x)(\Delta p) \geq 1/2$$

Determinism is lost: strictly speaking, no phase space!

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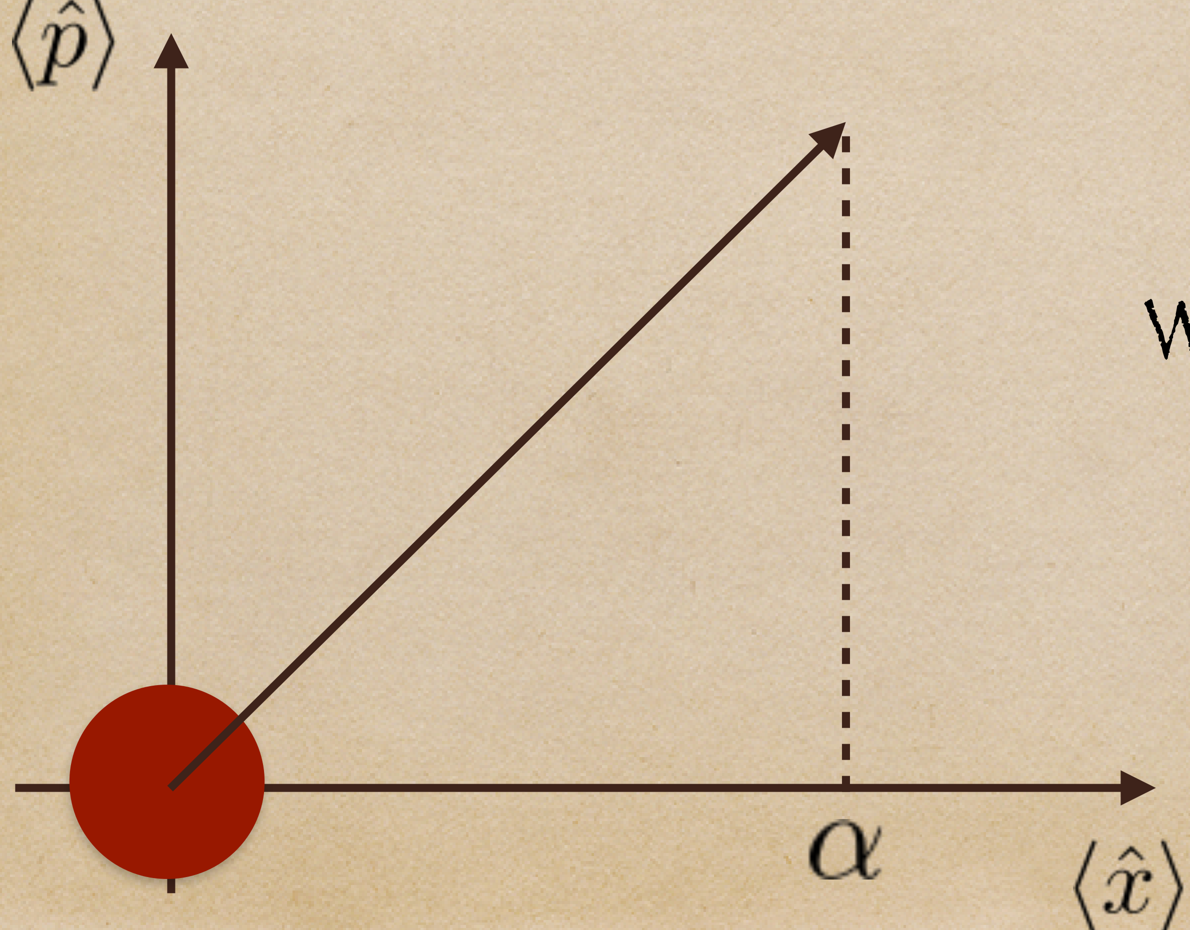
Yet, very useful...

1-to-1 correspondence between states and phase-space description

$$\rho = \frac{1}{\pi^N} \int \cdots \int \chi(\alpha_1, \ldots, \alpha_N) \left(\otimes_{j=1}^{N} \hat{D}_j^\dagger(\alpha_j) \right) d^2\alpha_1 \cdots d^2\alpha_N$$



$$\hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}$$

Weyl displacement operator

Yet, very useful...

1-to-1 correspondence between states and phase-space description

$$\rho = \frac{1}{\pi^N} \int \cdots \int \chi(\alpha_1, \ldots, \alpha_N) \left(\otimes_{j=1}^{N} \hat{D}_j^\dagger(\alpha_j) \right) d^2\alpha_1 \cdots d^2\alpha_N$$

$$\mathrm{Tr}[\hat{D}(\alpha)\hat{D}(\beta)] = \pi \delta^2(\alpha - \beta)$$

$$\chi(\alpha_1, \ldots, \alpha_N) = \mathrm{Tr}[\rho(\otimes_{j=1}^{N} \hat{D}_j(\alpha_j))]$$

Weyl characteristic function

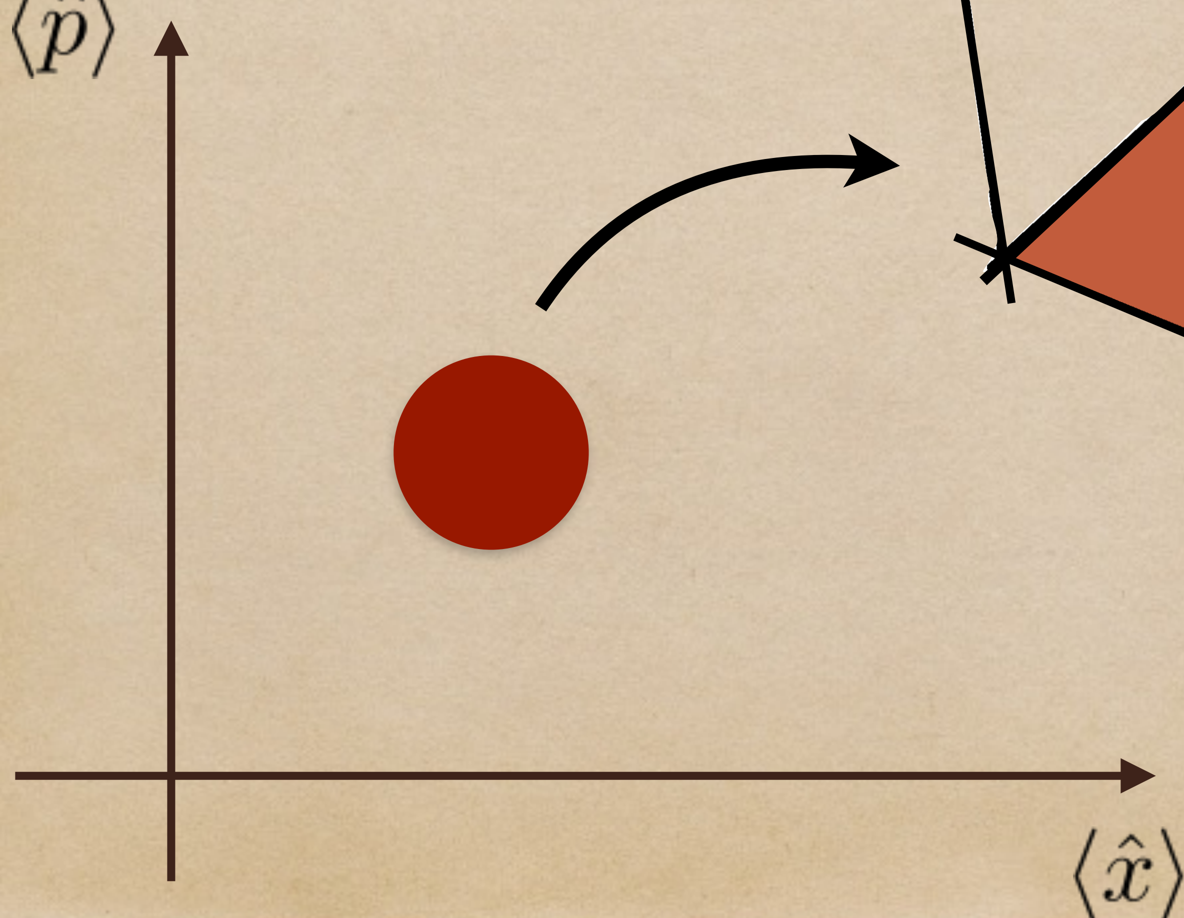
$$W_\rho(\xi_1 \cdots \xi_N) = \mathcal{F}^{\otimes N}_{\xi_1 \cdots \xi_N}[\chi(\alpha_1, \cdots, \alpha_N)]$$

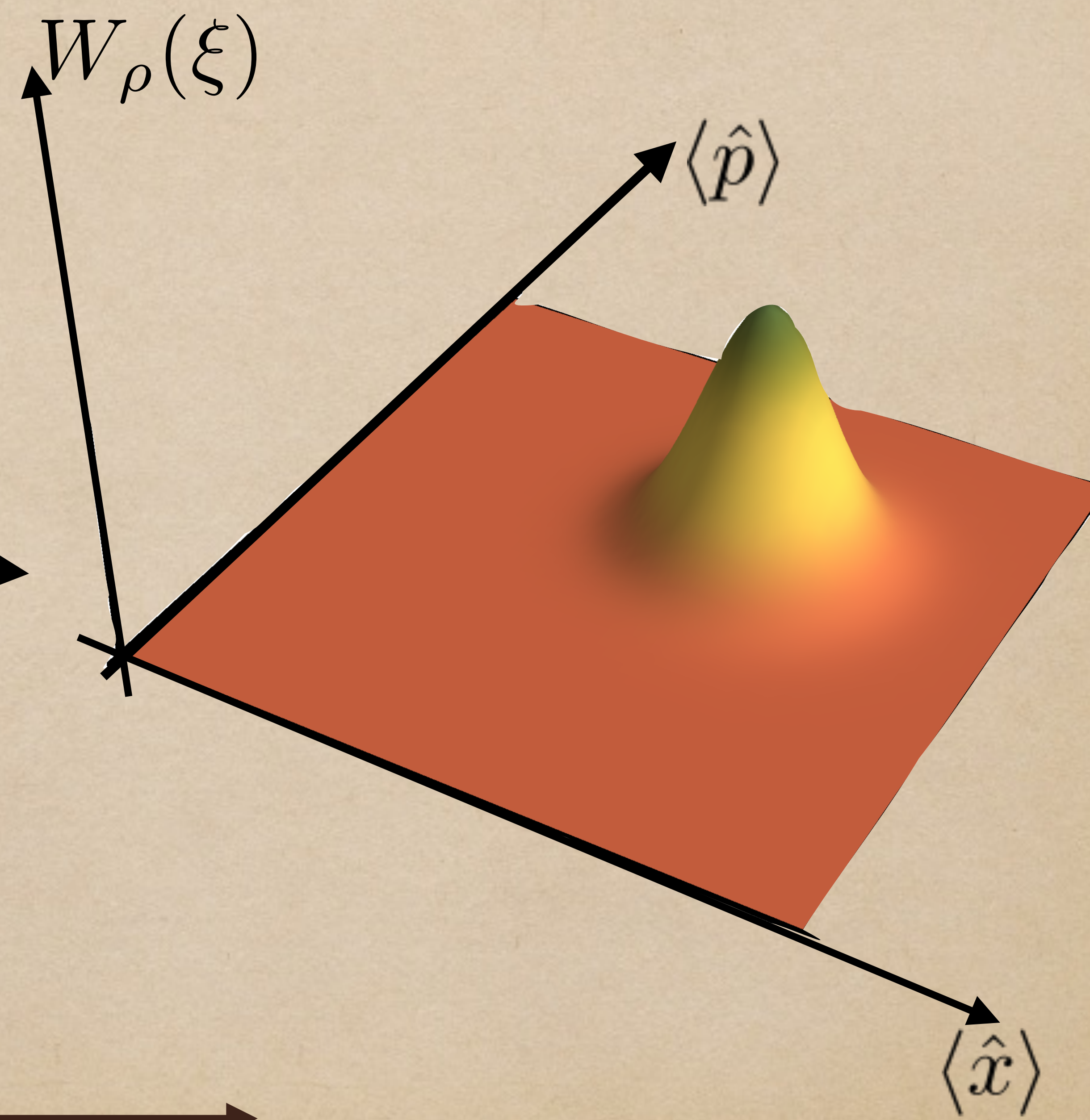
Wigner function

Yet, very useful...

$$\langle \hat{x} \rangle = \sqrt{2}\xi_r$$

$$\langle \hat{p} \rangle = \sqrt{2}\xi_i$$





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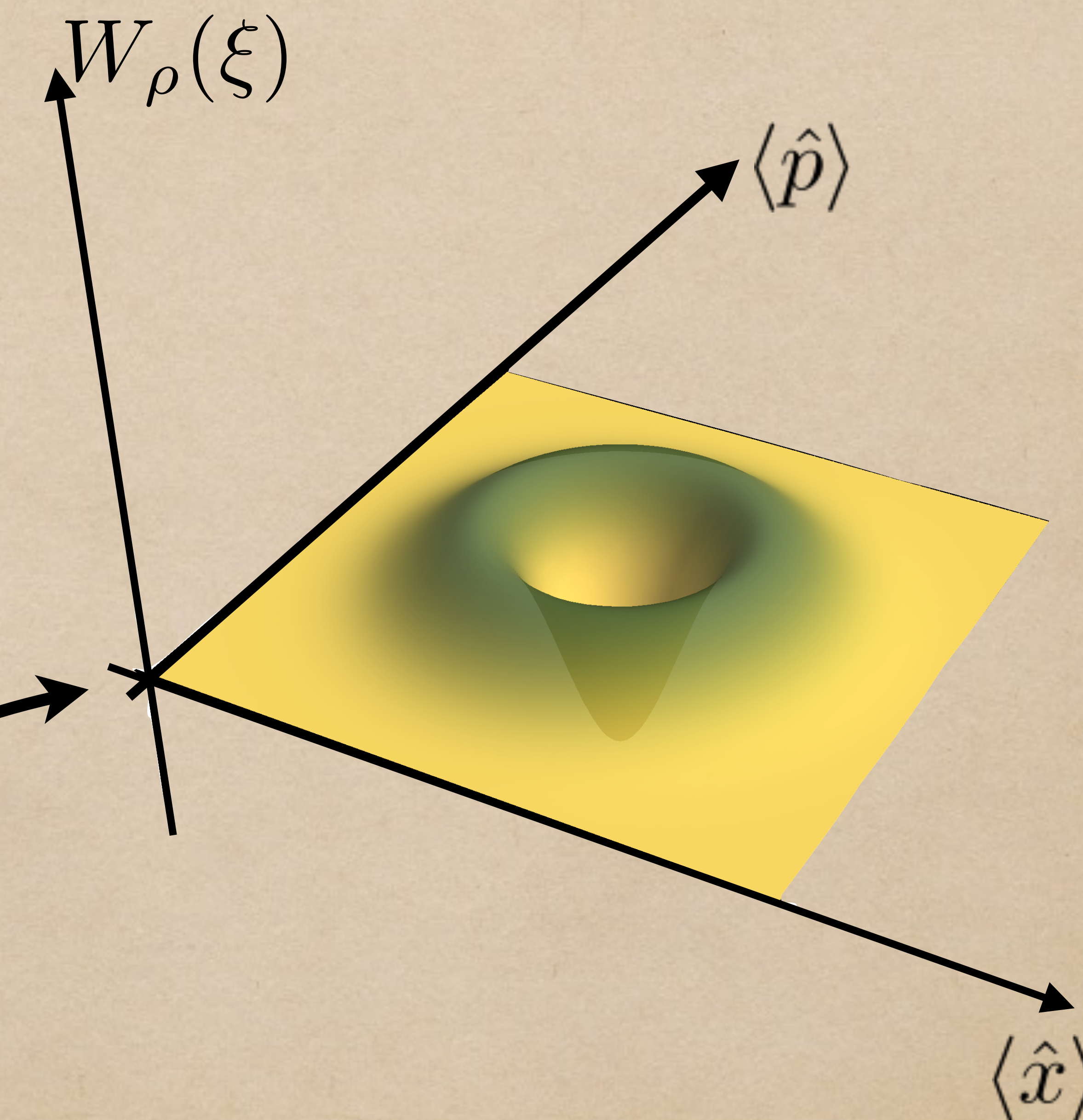
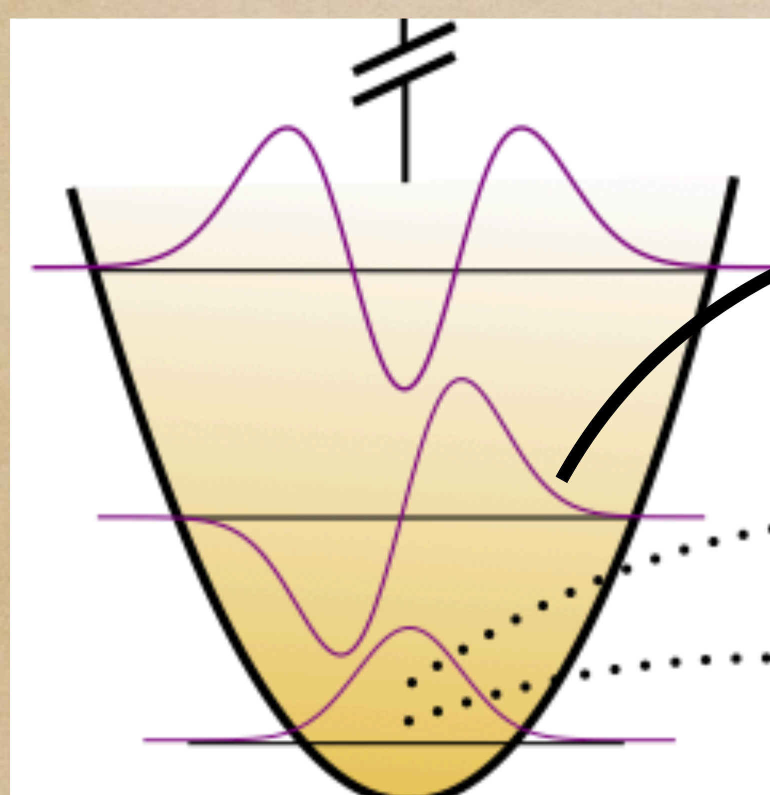
Yet, very useful...

$$\langle \hat{x} \rangle = \sqrt{2}\xi_r$$

$$\langle \hat{p} \rangle = \sqrt{2}\xi_i$$





Yet, very useful…

$$\langle \hat{x} \rangle = \sqrt{2}\xi_r$$

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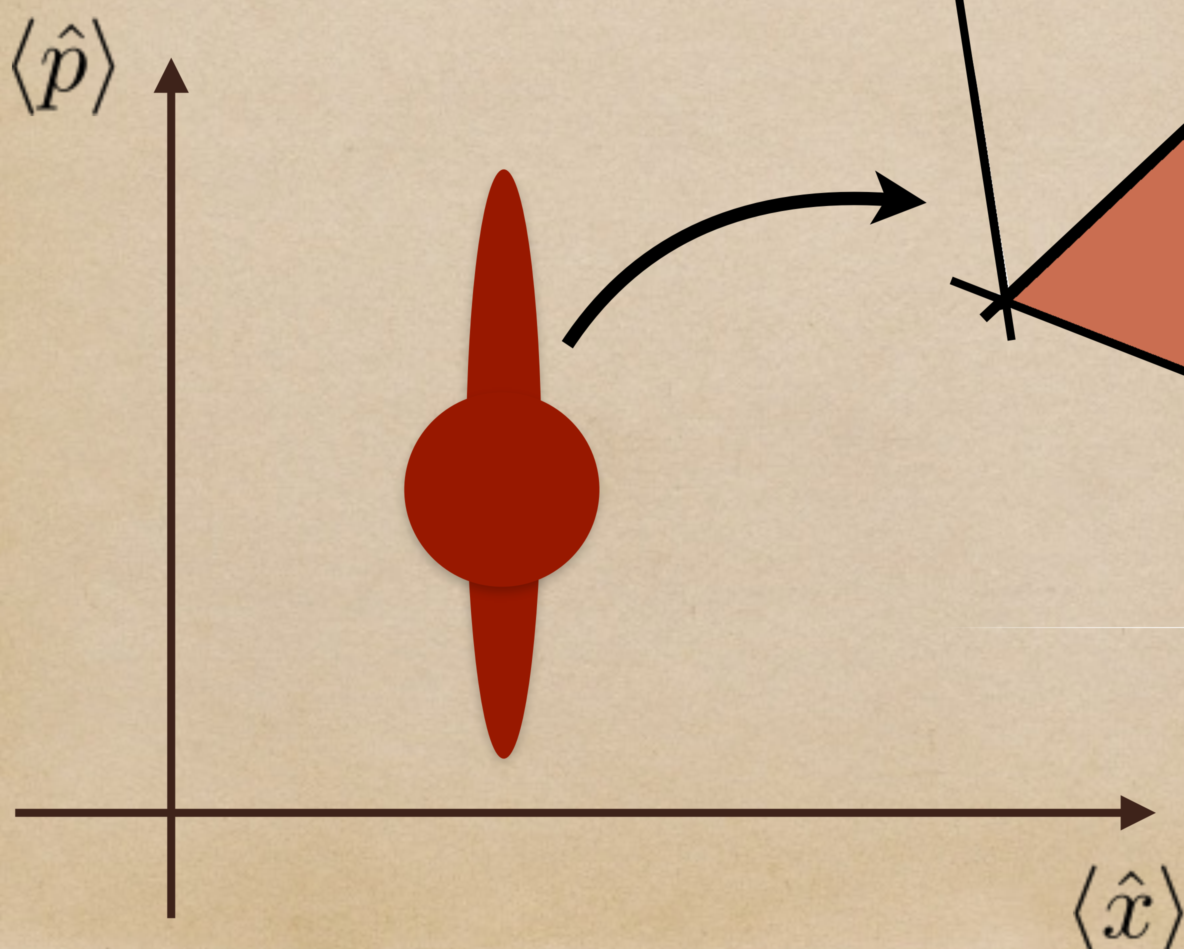
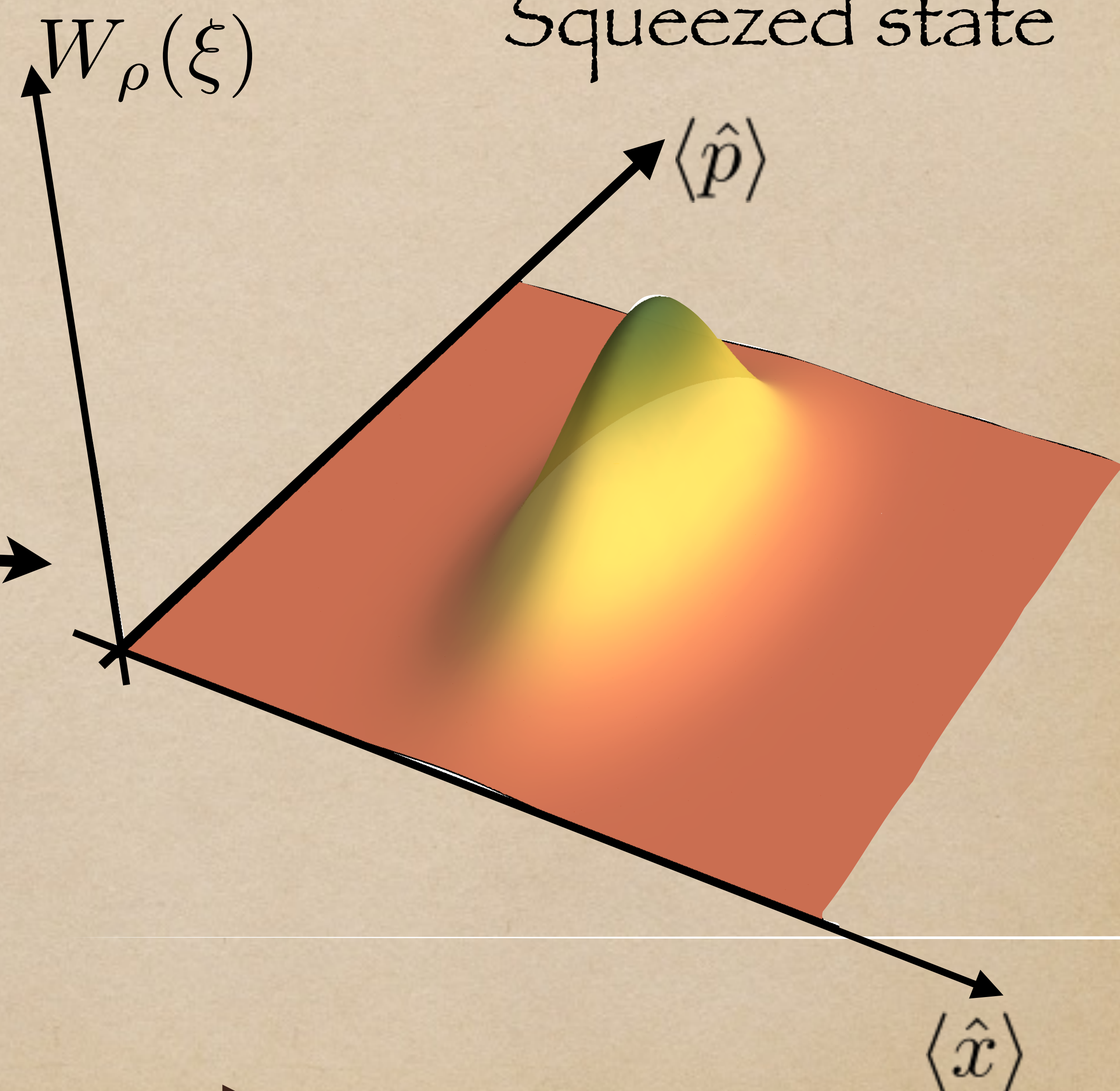
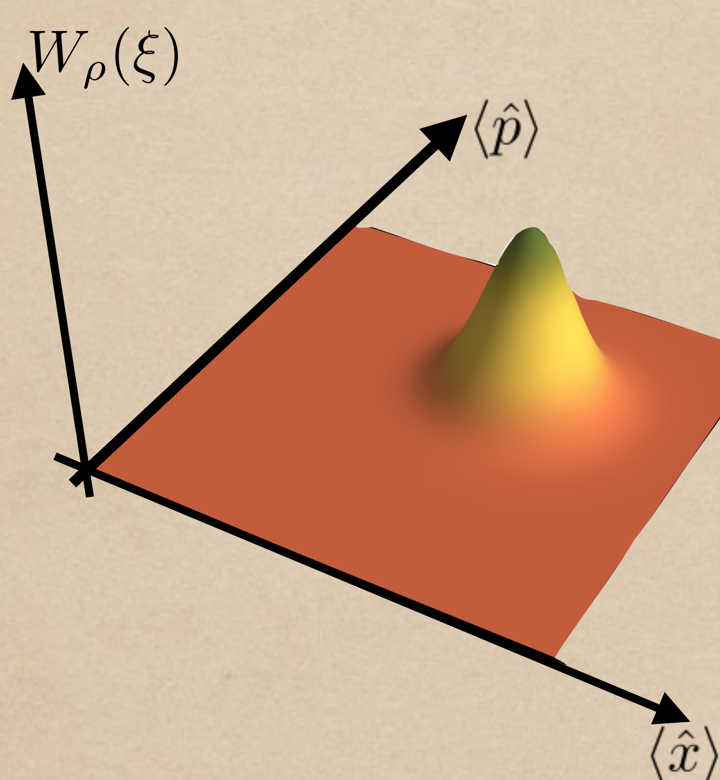
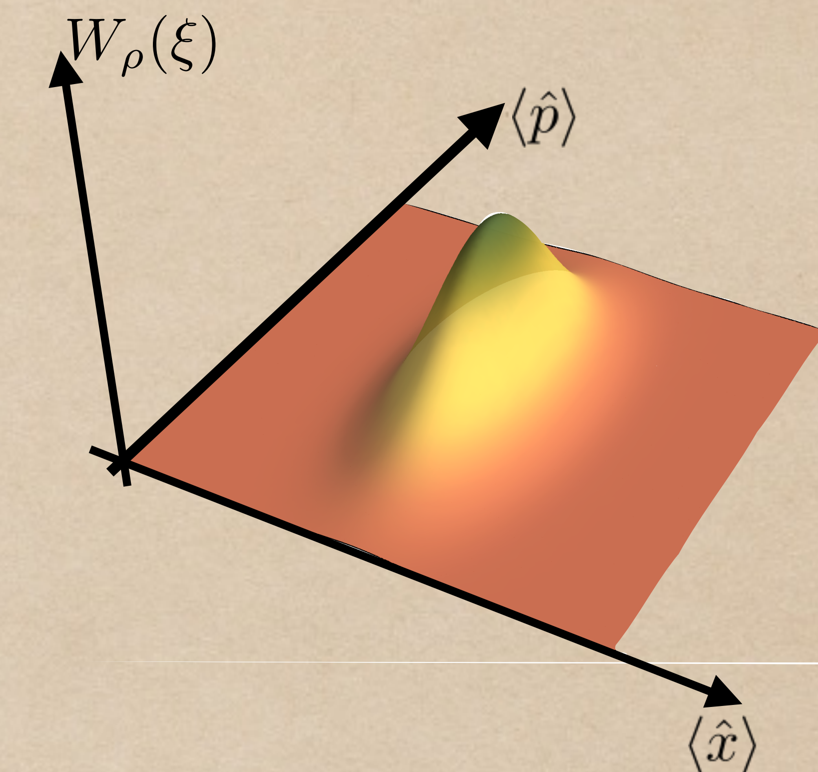


Squeezed state



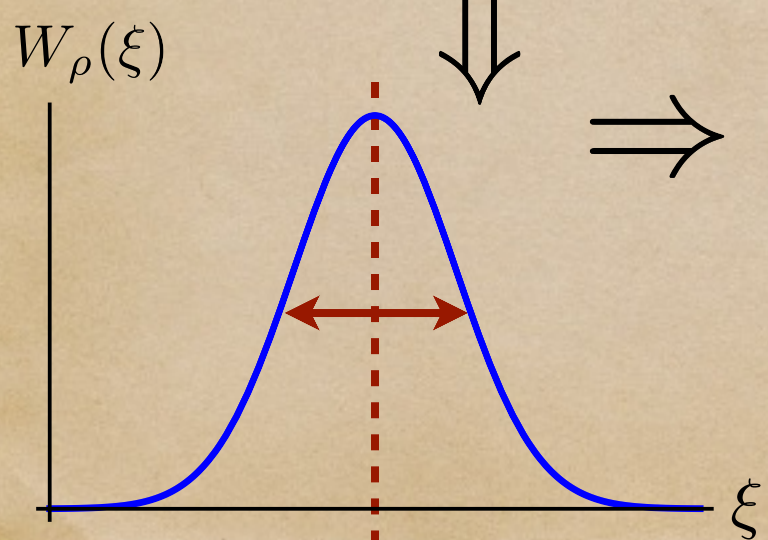
Gaussian states





Gaussian states:= states with a Gaussian Wigner function

$\Downarrow$



$\Rightarrow$ 1st and 2nd moments of phase-space variables

$$\mathbf{v} = (x_1, p_1, x_2, p_2, \ldots, x_N, p_N)$$

$$\sigma_{ij} = \langle\{v_i, v_j\}\rangle - 2\langle v_i\rangle\langle v_j\rangle$$

Gaussian states

$$\sigma_{ij} = \langle\{v_i, v_j\}\rangle - 2\langle v_i\rangle\langle v_j\rangle$$ Covariance matrix

$$W_\rho(\xi) = \frac{e^{-(\xi-\langle\xi\rangle)^{\mathsf{T}}\sigma^{-1}(\xi-\langle\xi\rangle)}}{\pi^N\sqrt{\det\sigma}}$$

True in general $\rho \Leftrightarrow W_\rho(\xi) \Leftrightarrow \sigma$ Only Gaussian states

	Hilbert	Phase-space
dimension	∞	$2N$
descriptor	ρ	$\sigma/W_\rho(\xi)$
structure	$\otimes$	$\oplus$
evolution	$\hat{U}$	?

Gaussian states

$$\hat{U}\rho\hat{U}^\dagger \Leftrightarrow S\sigma S^\mathsf{T}$$

Symplectic transformation

$$\Sigma = \oplus_{j=1}^{N} i\sigma_y \Rightarrow S^\mathsf{T}\Sigma S = \Sigma$$

(it's a group)

Symplectic matrix

	Hilbert	Phase-space
dimension	∞	$2N$
descriptor	ρ	$\sigma/W_\rho(\xi)$
structure	$\otimes$	$\oplus$
evolution	$\hat{U}$	$\mathbb{S}$

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Gaussian states

$$\rho \begin{cases} \mathrm{Tr}[\rho] = 1, \\ \text{Hermitian}, \\ \rho \geq 0 \end{cases} \qquad \sigma \begin{cases} \frac{1}{\pi}\int d^2\xi W_\rho(\xi) = 1, \\ \text{Real}, \\ \boxed{\sigma + i\Sigma \geq 0} \end{cases}$$

Heisenberg-Robertson condition

$$\mathrm{Eig}[\rho_{12\ldots N}] = \{e_j\}$$ via global unitary transformation

$$\sigma = \begin{pmatrix} \sigma_1 & \gamma_{12} & \cdots & \gamma_{1N} \\ \gamma_{12}^{\mathsf{T}} & \sigma_2 & \ddots & \vdots \\ \vdots & \ldots & \ldots & \vdots \\ \gamma_{1N}^{\mathsf{T}} & \gamma_{2N}^{\mathsf{T}} & \cdots & \sigma_N \end{pmatrix} \xrightarrow{S} \mathrm{Diag}[\mu_1, \mu_1, \ldots, \mu_N, \mu_N]$$

symplectic eigenvalues $\quad \{\mu_k\} = \mathrm{Eig}|i\Sigma\sigma|$

Gaussian states

Peres-Horodecki separability condition

ρ_{AB} is separable $\Leftrightarrow \rho_{AB}^{\mathsf{PT}}$

$2 \otimes 2$

$2 \otimes 3$

$\infty \otimes \infty$ (Gaussian)

$$\rho_{AB} \text{ separable} \Leftrightarrow \sigma^{\mathsf{PT}} + i\Sigma \geq 0$$

$$\Downarrow$$

$$\min \boxed{\text{Eig}|i\Sigma\sigma^{\mathsf{PT}}|} \geq 1$$

symplectic eigenvalues

$$\sigma^{\mathsf{PT}} = P\sigma P$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

momentum inversion